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MATHEMATICS magazine

MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

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Second Annual

MATHEMATICS WORKSHOP

SPONSORS:

The University of Arkansas and The Arkansas Council of Teachers of Mathematics.

DIRECTOR:

Professor Davis P. Richardson, Department of Mathematics, University of Arkansas.

DATES:

June 28 - July 2, 1954.

PURPOSE:

The purpose of the Workshop is to bring together high school and elementary teachers of mathematics to study problems of common interest and to work for the improvement of instruction in mathematics. For further information write: Mathematics Department, University of Arkansas, Fayetteville, Arkansas.

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HIGH SCHOOL MATHEMATICS TEACHERS

AT:

University of Washington, Seattle, Washington.

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DATES:

July 26 - August 20, 1954.

PURPOSE:

The purpose of this conference is to aid in the improvement of the teaching of high school mathematics through the joint efforts of university and school mathematicians. The conference will emphasize modern viewpoints in the teaching of algebra and geometry, will discuss proposed revisions of the high school mathematics curriculum which suggest the inclusion of topics such as: logic, analytic geometry, calculus and statistics, and will offer opportunities for teachers to see mathematics at work in various Seattle industries.

The conference has been arranged so that teachers planning to attend the summer meeting of the National Council of Teachers of Mathematics to be held in Seattle August 22-25 may come a month earlier to participate in the conference program.

ARISTOTLE D. MICHAL

1899 - 1953

D. H. Hyers

On June 14, 1953, Aristotle D. Michal, Professor of Mathematics, California Institute of Technology and one of the principal editors of this Magazine, died of heart disease.

Born in Smyrna, Asia Minor in 1899 of Greek parents, Michal came to the United States when a youth of twelve years to become a permanent resident of this country.

After graduating from Clark University at Worcester, Massachusetts in 1920, he stayed on as a graduate fellow, receiving a Master's degree in 1921. From 1921-24 he studied at Rice Institute, Houston, Texas, where he was influenced particularly by Griffith C. Evans and his work on functionals. The year 1924 was an eventful one for Michal, for in it he became an American citizen, received the doctorate from Rice Institute and married Luddye Kennerly of Houston, Texas.

After teaching at Rice as an instructor during 1923-25, he was awarded a National Research Fellowship for the next two years, which he spent at Harvard, Chicago and Princeton Universities.

In 1927 Dr. Michal was appointed Assistant Professor of Mathematics at the Ohio State University. Two years later he joined the faculty of the California Institute of Technology as Associate Professor of Mathematics. He was promoted to a Professorship in 1938.

Michal was a very prolific researcher and author, as the appended list of his publications will testify. Here we shall be able only to mention some of the highlights of his work which extended over a period of some thirty years.

During 1924-26 his research was concerned chiefly with groups of functional transformations (e.g. Fredholm or Volterra integral transformations) and their differential or integro-differential invariants. Beginning with the joint paper [6] with T. Y. Thomas on differential invariants, the research takes on a more geometrical flavor. Not content with the study of tensor analysis, differential geometries and differential invariants in n - dimensional spaces, Michal began to develop these theories for function spaces in [10] (1928). In place of an n - dimensional contravariant vector with components y^i, $i = 1, \cdots, n$ he considered an infinite dimensional "vector" with components y^α, where α varies continuously over a fixed interval $a \leq \alpha \leq b$ and where y is a continuous function of α. In place of the quadratic metric form $ds^2 = \sum g_{ij} dy^i dy^j$ for an n - dimensional Riemannian geometry, he considered quadratic functional forms, such as

$$\int_a^b\int_a^b g_{\alpha\beta}[y]\,dy^\alpha dy^\beta\,d\alpha d\beta \;+\; \int_a^b (dy^\alpha)^2 d\alpha$$

as the fundamental form for a Riemannian functional geometry. Non-Riemannian functional geometries were also developed in [10, 11, 12, 13].

In the early thirties, Michal eagerly embraced the new functional analysis being developed by Frechet, Banach and others. The more abstract approach afforded not only a generalization but also a simplification and a more fundamental approach to Michal's theories of differential geometry in function space. However, the new approach made necessary research on such topics as multilinear forms, differentials and differential equations in abstract vector spaces, polynomials and analytic functions in Banach spaces, and normed rings.* Research was carried out by Michal and his many students in all of these subjects over a long period of years, as well as on abstract differential geometry, which was the initial motivation. To get an idea of what was accomplished in the latter field up to 1938, the reader is referred to [47], which is a report given in an hour's invited address before the American Mathematical Society in April, 1938.

Since then the work on differential geometry in abstract spaces has developed along several lines. Abstract projective geometry was the subject of four papers [48, 49, 53, 55]. An interesting relationship between vibration problems for continuous media and geodesics in certain infinitely dimensional Riemannian spaces was worked out for two special cases in [59] and [63]. In [66] Michal returned to the study of the special case of abstract Riemannian and Hermitian spaces of constant curvature. Such infinite dimensional Hermitian spaces appear to be of importance as universal enveloping spaces into which certain types of finite dimensional Hermitian spaces may be imbedded.**

A differential (of the Fréchet type) for functions with arguments and values in a linear topological space was defined by Michal and Paxson in 1936 [30, 34]. Later Michal gave several other more satisfactory definitions [45, 47] and even extended the notion of a differential to functions defined on topological abelian groups [57, 58].

The subject of differential equations in abstract spaces was developed in a series of papers extending over many years [29, 31, 38, 60, 67, 69, 73, 74, 76] and culminating in the book, *Differential Equations in Abstract Spaces, with Applications* which is still to be published. Although others had considered *ordinary* differential equations for functions of a *real* variable whose values lie in a Banach space, Michal and

*The first definition and consideration of normed rings (now often called Banach algebras) was probably given by Michal and Martin in [27].

**See S. Bochner, Bull. Am. Math. Soc. 53 : 179-195 (1947), especially p. 193.

his student Elconin [29] in 1935 appear to be the first to have considered "Pfaffian" equations in which both independent and dependent variables lie in Banach spaces, and to prove existence theorems for such equations. They were also the first to develop the beginnings of a theory of Lie groups for transformation $y = f(x, a)$ where x, y and the parameter a all lie in Banach spaces [35].

After this beginning, Michal returned several times to the study of continuous transformation groups in abstract spaces and in abstract differential geometries [70, 72]. In particular, groups of motions for abstract Euclidean spaces were taken up in [39], and for abstract Riemannian spaces in [68].

Studies of the solutions of differential equations as analytic functionals of the coefficient functions were carried out in [69] and [76].

We can mention here only very briefly the important subjects of polynomials and analytic functions in Banach spaces, to which Michal and his students have made fundamental contributions. Michal and his students I. E. Highberg and R. S. Martin developed a theory of polynomials and of abstract power series back in the thirties, while Angus E. Taylor built a theory of analytic functions in complex Banach spaces on the concept of the Gateaux differential.*

Michal returned to the study of polynomials and analytic functions in both real and complex spaces in [62] and in [75]. Polygenic functions in general analysis were taken up in [52].

During his career as a Professor, Michal was responsible for no less than twenty eight Ph.D.'s. He was extremely skillful and successful at guiding graduate students into research. I recall that when I was a graduate student at the California Institute in the thirties, there were about a dozen students attending his stimulating seminars and most of these later wrote their dissertations and received their degrees under Michal's direction.

His death is a great loss not only to American mathematics in general but particularly to his many students and friends, and to the editors and readers of the Mathematics Magazine, to which he contributed so much in the last six years.

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1. Matrix and Tensor Calculus with Application to Mechanics, Elasticity and Aeronautics. John Wiley and Sons, N.Y. (1947).
2. Differential Equations in Abstract Spaces with Applications (to be published).

*Ann. R. Scuola Norm. Sup. Pisa (2) v.6 : 277-292 (1937). For further developments in this field see for example, A. E. Taylor, Bull. Am. Math. Soc. v.49 : 652-669 (1943) and Hille's book, *Functional Analysis and Semi-Groups*, 1948, Chapter IV.

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DISCONTINUITIES IN COMPRESSIBLE FLUID FLOW

N. Coburn

I

INTRODUCTION

The theory of discontinuities in flows of compressible fluids has been treated by many authors beginning with B. Riemann and his contemporaries and culminating in the elegant theory of J. Hadamard[1] and the excellent texts by Courant-Hilbert[2] and Courant-Friedrichs[3]. Hadamard's theory is based on the Lagrangian form of the equations of motion, and Courant-Friedrichs use moving coordinates to study shock and contact discontinuities. In the following sections we shall show that by following a method developed by R. K. Luneberg[4] for electromagnetic theory, one may obtain a unified theory of discontinuities in terms of Eulerian variables.

II

THE GENERAL THEORY

We consider a four-dimensional Euclidean space-time manifold, E_4. The time variable will be denoted by t, and the space variables by x^λ, $\lambda = 1,2,3$. Sometimes, we may wish to consider both the time and space variables simultaneously. Then, we shall use the notation x^j, $j = 0,1,2,3$, where x^0 is the time variable t, and x^1, x^2, x^3 denote the space variables.

For the purposes of this paper, it will be sufficient to assume that the space variables determine Cartesian orthogonal coordinate systems in ∞^1 Euclidean three-spaces E_3 in E_4. The coordinate lines along which only the time variable t varies will be assumed to be straight lines which are orthogonal to these $\infty^1 E_3$. An analogous situation in ordinary Euclidean three-space is obtained by introducing orthogonal Cartesian coordinates. The coordinate lines along which only z varies would correspond in our case to the coordinate lines along which only t varies; and the variables x, y would correspond to our space variables x^λ, $\lambda = 1,2,3$.

1. J. Hadamard, Propagation des Ondes, Chelsea Publishing Co., N. Y., 1949.
2. R. Courant and D. Hilbert, Methoden der Mathematischen Physik, Interscience Publishers, N. Y., Vol. II.
3. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience Publishers, N. Y., 1948.
4. R. K. Luneberg, Asymptotic Development of Steady State Electromagnetic Fields, N.Y.U., Mathematics Research Group, Research Report No. EM 14, July, 1949.

The permissible group of coordinate transformations is given by

$$(2.1) \qquad {}'x^0 = x^0, \qquad {}'x^\lambda = \alpha^\lambda_\mu x^\mu, \qquad \lambda,\mu = 1,2,3,$$

where the constants α^λ_μ determine an orthogonal matrix with determinant, +1. As is well known, for the case of the special orthogonal group (2.1), the theory of Cartesian tensors[5] may be used. In this case, there is no difference between covariant and contravariant quantities since both transform by the same law. However, we shall use both of these quantities in order to work with the Einstein summation convention, in which only repeated covariant and contravariant indices are summed.

Now, we consider the metric tensor in our class of preferred Cartesian orthogonal coordinate systems. If we denote the covariant components of this tensor by g_{jk}, $j,k = 0,1,1,3$, then we find that

$$(2.2) \qquad g_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

An alternative description of this tensor is given by separating out the time and space components. That is, we say that the tensor, g_{jk}, consists of the time-like tensor with component

$$g_{00} = 1$$

and of the space-like tensor, $g_{\lambda\mu}$, $\lambda,\mu = 1,2,3$, with components

$$(2.3) \qquad g_{\lambda\mu} = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ 1 & \text{if } \lambda = \mu. \end{cases}$$

Corresponding to the description (2.2), we may introduce the space-time covariant components, w_j, of the contravariant vector, w^j, by writing

$$w_j = g_{jk}\, w^k.$$

Similarly, the use of the space-like metric tensor, $g_{\lambda\mu}$, enables us to introduce the space covarient components, v_λ, of the contravariant vector, v^λ, by means of

$$(2.4) \qquad v_\lambda = g_{\lambda\mu}\, v^\mu.$$

Actually, we shall use (2.4) in the remainder of our theory. The reason for this lies in the fact that Newtonian mechanics is a theory of a (1 + 3) dimensional manifold and not of a four dimensional space-time manifold. The only exception to this rule for using covariant indices will occur in differentiation. That is, we shall write

5. H. Jeffreys, Cartesian Tensors, Cambridge University Press, 1931.

(2.5) $$\frac{\partial}{\partial t} = \partial_0, \qquad \frac{\partial}{\partial x^\lambda} = \partial_\lambda, \qquad \lambda = 1,2,3,$$

or more generally

(2.6) $$\frac{\partial}{\partial x^j} = \partial_j, \qquad j = 0,1,2,3.$$

It should be noted that, for the metric tensors (2.2) and (2.3), the Christoffel symbols vanish and covariant derivatives reduce to partial derivatives. Also, the range of the Greek indices is always 1,2,3; the range of the Latin indices is 0,1,2,3.

In the above preferred coordinate systems, the basic equations of hydrodynamics of a compressible non-viscous fluid, which consist of the equations of motion, continuity, and energy, respectively, may be written as[6]

(2.7) $$\frac{\partial}{\partial t}(\rho v_\lambda) + \partial_\mu(\rho v^\mu v_\lambda + p\,\delta^\mu_\lambda) = 0,$$

(2.8) $$\frac{\partial \rho}{\partial t} + \partial_\mu(\rho v^\mu) = 0,$$

(2.9) $$\frac{\partial}{\partial t}[\rho(\frac{q^2}{2} + e)] + \partial_\mu[\rho v^\mu(\frac{q^2}{2} + h)] = 0,$$

where δ^μ_λ is the Kronecker delta tensor, ρ denotes the density, p the pressure, v^λ the velocity vector, q the magnitude of this vector, e the specific internal energy, h the specific enthalpy. The equations (2.7) through (2.9) form a non-linear system of five partial differential equations for five unknowns (the two independent thermodynamic variables such as ρ, p, and the three velocity components, v^λ). For polytropic gases, the two independent thermodynamic variables are usually chosen to be ρ, T (the absolute temperature) or ρ, c (the local sound speed). Then, it follows that[7]

$$e = c_v T, \qquad h = \frac{\gamma}{\gamma-1} RT = \frac{c^2}{\gamma-1}, \qquad p = R\rho T,$$

where c_v, γ, R are constants with thermodynamical significance.

In order to initiate our study of discontinuities, we introduce the hypersurface (or lower dimensional manifold) in space-time along which the discontinuities occur. We denote this manifold by

6. See reference 3, pp. 15, 16, formulas (7.09.2), (7.08.2), (8.02.2). These equations follow from the usual forms of the equations of motion, continuity, and energy.

7. See reference 3, pp. 7, 17. Note that the relation $p = R\rho T$ is gas law of Boyle and Gay-Lussac.

$$\phi_j(t,x^\lambda) = c_j,$$

$$j = 0, \quad \text{or} \quad j = 0,1, \quad \text{or} \quad j = 0,1,2, \quad \text{or} \quad j = 0,1,2,3,$$

where c_j are constants. It should be noted that $\phi_j(t,x^\lambda)$ represents a function of t and x^1, x^2, x^3. Similar abbreviations are used throughout this paper. If this system consists of only one equation ($j=0$), then the discontinuity manifold defines a hypersurface. Similarly, if the system consists of two equations ($j=0,1$) then the equations define a surface; if this system contains three equations ($j=0,1,2$) then the discontinuity lies along a curve; and if this system consists of four equations ($j=0,1,2,3$) then the discontinuity consists of a finite number of points. In any case, the vector fields for the various types of j,

$$\frac{\partial\phi_j}{\partial t}, \quad \frac{\partial\phi_j}{\partial x^\lambda}$$

determine vectors normal to the discontinuity manifold[8]. Further, the unit normal vectors of this manifold are determined by

$$(2.10) \qquad \begin{aligned} n_0 &= \frac{\partial\phi}{\partial t} \Big/ \left[\left(\frac{\partial\phi}{\partial t}\right)^2 + g^{\lambda\mu}\frac{\partial\phi}{\partial x^\lambda}\frac{\partial\phi}{\partial x^\mu}\right]^{1/2} \\ n_\alpha &= \frac{\partial\phi}{\partial x^\alpha} \Big/ \left[\left(\frac{\partial\phi}{\partial t}\right)^2 + g^{\lambda\mu}\frac{\partial\phi}{\partial x^\lambda}\frac{\partial\phi}{\partial x^\mu}\right]^{1/2} \end{aligned}$$

where the subscript j has been omitted to gain simplicity in writing.

If we integrate (2.7) through (2.9) over some region of E_4 which is bounded by a closed hypersurface S_3, we find through use of the Gauss formula[9] (also called the Stokes formula).

$$(2.11) \qquad \int_{S_3} [n_0\rho v_\lambda + n_\mu(\rho v^\mu v_\lambda + p\,\delta^\mu_\lambda)]\,d\tau = 0,$$

$$(2.12) \qquad \int_{S_3} [n_0\rho + n_\mu\rho v^\mu]\,d\tau = 0,$$

$$(2.13) \qquad \int_{S_3} \left[n_0\rho\left(\frac{q^2}{2} + e\right) + n_\mu\rho v^\mu\left(\frac{q^2}{2} + h\right)\right] d\tau = 0.$$

8. Here as elsewhere in the paper the symbols $\left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x^\lambda}\right)$ represent the vector with components $\left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^2}, \frac{\partial\phi}{\partial x^3}\right)$.

9. J. A. Schouten and D. J. Struik, Einführung in die Neueren Methoden der Differentialgeometrie, Vol. I, P. Noordhoff, N. V.- Groningen, Batavia, p. 130, formula (11.82).

The vector with components n_0, n_α ($\alpha = 1,2,3$) represents the unit normal to S_3. We shall show that *the above equations are the fundamental equations for the study of those manifolds along which discontinuities in* ρ, p, *and* v^λ *propagate—the so-called shock*[10] *and contact manifolds.*

To determine the fundamental equations for the study of those manifolds along which discontinuities in the *derivatives of* ρ, p, and v^λ propagate—the characteristic manifolds—we proceed in a slightly different manner. By differentiation of (2.7) through (2.9), we obtain

$$\partial_j \frac{\partial}{\partial t} (\rho v_\lambda) + \partial_j \partial_\mu (\rho v^\mu v_\lambda + p\,\delta^\mu_\lambda) = 0,$$

$$\partial_j \frac{\partial \rho}{\partial t} + \partial_j \partial_\mu (\rho v^\mu) = 0,$$

$$\partial_j \frac{\partial}{\partial t} [\rho(\frac{q^2}{2} + e] + \partial_j \partial_\mu [\rho v^\mu (\frac{q^2}{2} + h)] = 0.$$

Assuming that the above functions possess continuous second derivatives, we may interchange the order of differentiation and write

$$\partial_j \partial_k\, v^\lambda = \partial_k \partial_j\, v^\lambda; \quad j,k = 0,1,2,3; \quad \lambda,\mu = 1,2,3; \quad \text{etc.}$$

If the derivatives in the previous equations are treated in this manner and then Gauss's theorem is applied to the resulting equations, we obtain

$$\text{(2.14)} \qquad \int_{S_3} [n_0 \partial_j (\rho v_\lambda) + n_\mu \partial_j (\rho v^\mu v_\lambda + p\,\delta^\mu_\lambda)]\, d\tau = 0,$$

$$\text{(2.15)} \qquad \int_{S_3} [n_0 \partial_j \rho + n_\mu \partial_j (\rho v^\mu)]\, d\tau = 0,$$

$$\text{(2.16)} \qquad \int_{S_3} \{ n_0 \partial_j [\rho(\frac{q^2}{2} + e)] + n_\mu \partial_j [\rho v^\mu (\frac{q^2}{2} + h)]\}\, d\tau = 0.$$

The equations (2.14) *through* (2.16) *are the fundamental equations for the study of discontinuities which propagate along characteristics.*

Since the theorem of Gauss was used in deriving the above systems of equations, the region of E_4 bounded by the closed hypersurface S_3 cannot contain points at which ρ, p, v^λ, etc., and their derivatives are discontinuous. This is a very severe limitation on the formulas. In order

10. Shocks are irreversible thermodynamic processes. Hence, one must remember that an additional principle is needed; enthropy always increases across a shock. This result will not concern us in this paper but is needed for a *detailed* study of the shock conditions of section 4 (see reference 3, p. 137).

to extend the above formulas to the case where E_4 contains points of discontinuity, three procedures are available: (1) one may attempt to apply the previous integral relations to the case where S_3 intersects a discontinuity manifold and "squeeze down" to the discontinuity manifold; (2) one may postulate that the above two systems of equations are the fundamental equations of fluid flow and apply whenever the integrals exist; (3) *one may postulate that only equations* (2.11) *through* (2.13) *are the fundamental equations of fluid flow and apply whenever the integrals exist.* The first procedure involves the use of complicated limit processes; the second procedure is logically sound and easy to apply but *appears to be redundant to the present author.* For these reasons we shall use the third procedure.

We postulate that the system of equations (2.11) *through* (2.13) *are the fundamental equations of compressible fluid flow and apply whenever the integrals involved exist.*[11] First, we note that if ρ, p, v^λ, etc., and their derivatives are continuous in a region of space-time and we apply the divergence theorem of Gauss, then the fundamental equations reduce to the equations of motion (2.7), the continuity relation (2.8), and the energy relation (2.9), respectively. Furthermore, the fundamental equations are applicable even when ρ, p, v^λ, etc., are discontinuous, provided that the integrals over the closed hypersurface S_3 exist. This means that our fundamental system of hydrodynamical equations can be applied to the case where the closed hypersurface S_3 intersects a manifold S_3 along which ρ, p, v^λ, etc., have finite jumps. For in this case, ρ, p, v^λ, etc., are not defined along the surface S_2 of intersection of S_3, S_3. However, the integrals (2.11) through (2.13) exist and are independent of which finite values are assigned to ρ, p, v^λ, etc., along S_2. This is due to the fact that the points of discontinuity along S_2 *have zero three-dimensional measure*.[12]

We shall show that *the system* (2.14) *through* (2.16) *can be deduced from our fundamental equations* (2.11) *through* (2.13). To simplify the discussion we replace the fundamental equations by the single representative relation

$$\int_{S_3} [n_0 h_\lambda + n_\mu k^{\mu}_{\cdot\lambda}]\, dt = 0. \tag{2.17}$$

Thus, if (2.17) represents the equations of motion then the tensors h_λ, $k^{\mu}_{\cdot\lambda}$ reduce to

$$h_\lambda = \rho v_\lambda, \qquad k^{\mu}_{\cdot\lambda} = \rho v^\mu v_\lambda + p\delta^{\mu}_{\lambda}.$$

11. To be able to apply our theory to lower dimensional discontinuity manifolds, we must assume that S_3 may be replaced by a lower dimensional manifold.
12. S. Saks, Theory of the Integral, translated by L. C. Young, Hafner Pub. Co., N.Y., 1937, p. 23.

To represent (2.12), (2.13) we must replace h_λ by a scalar h, and $k^{\mu}_{\cdot\lambda}$ by a vector k^{μ}. However, we shall consider (2.17) as typical of all of our fundamental equations. First, we note that the arguments of h_λ, $k^{\mu}_{\cdot\lambda}$ are the coordinates of the Cartesian orthogonal system, x^λ and t. Now, we consider the one-parameter family of solutions of the representative equation (2.17):

$$'h_\lambda = h_\lambda(x^1+\xi,\ x^2,\ x^3,\ t), \quad 'k^{\mu}_{\cdot\lambda} = k^{\mu}_{\cdot\lambda}(x^1+\xi,\ x^2,\ x^3,\ t)$$

The existence of this one-parameter family of solutions follows directly from (2.17). That is, by use of the translation transformation of coordinates, $'x^1 = x^1 + \xi$, we see that

$$\int_{'S_3} [\,'n_0\, 'h_\lambda + 'n_\mu\, 'k^{\mu}_{\cdot\lambda}]\, d\tau = 0$$

where $'S_3$ is the translated hypersurface S_3 and $'n_0$, $'n_\mu$ are the components of the unit normal of $'S_3$. However, S_3 is arbitrary. Hence, we may drop the primes from the surface quantities in the above equation and obtain

$$\int_{S_3} [n_0\, 'h_\lambda + n_\mu\, 'k^{\mu}_{\cdot\lambda}]\, d\tau = 0.$$

If $'h_\lambda$, $'k^{\mu}_{\cdot\lambda}$ are continuous functions of ξ (or h_λ, $k^{\mu}_{\cdot\lambda}$ are continuous functions of x^1) and if these functions possess unique derivatives with respect to ξ along S_3 then we may differentiate the above relation under the integral sign. Further, allowing ξ to approach zero, we obtain

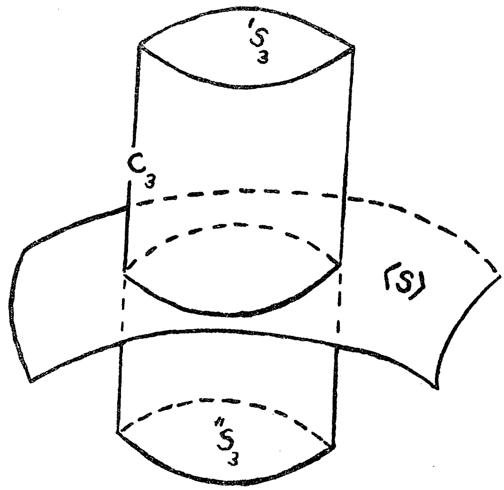
$$\int_{S_3} [n_0 \frac{\partial h_\lambda}{\partial x^1} + n_\mu \frac{\partial k^{\mu}_{\cdot\lambda}}{\partial x^1}]\, d\tau = 0.$$

However, even if $'h_\lambda$, $'k^{\mu}_{\cdot\lambda}$ (or h_λ, $k^{\mu}_{\cdot\lambda}$) do not possess unique derivatives on a surface S_2 of S_3 but only one-sided derivatives, the above relation remains valid. Evidently, one may replace the derivatives with respect to x^1 by derivatives with respect to x^2, x^3, t. The tensor form of the resulting equations is

$$\int_{S_3} [n_0 \partial_j h_\lambda + n_\mu \partial_j k^{\mu}_{\cdot\lambda}]\, d\tau = 0.$$

Thus, we have shown that the system (2.14) through (2.16) may be deduced from the fundamental system. In the remainder of this section we consider equations (2.17) as typical of either system.

To obtain the desired discontinuity relations, we introduce the manifold $\langle S \rangle$ along which the discontinuity occurs. Let S_3 be a closed hypersurface whose boundaries consist of: (1) two hypersurfaces $'S_3$, $''S_3$ which do not intersect $\langle S \rangle$; and (2) a cylindrical hypersurface C_3 which intersects $\langle S \rangle$.



The Hypersurface, $S_3 = 'S_3 + ''S_3 + C_3$

The typical equation (2.17) may be expanded into

$$\int_{'S_3} [n_0 h_\lambda + n_\mu k^{\mu}_{\cdot\lambda}]\, d\tau + \int_{''S_3} [n_0 h_\lambda + n_\mu k^{\mu}_{\cdot\lambda}]\, d\tau$$
$$+ \int_{C_3} [n_0 h_\lambda + n_\mu k^{\mu}_{\cdot\lambda}]\, d\tau = 0.$$

If we allow the length of the generators of the cylinder C_3 to shrink to zero *on each side* of $\langle S\rangle$, then $'S_3$, $''S_3$ shrink to $\langle S\rangle$ and the above integral reduces to

$$\int_{\langle S\rangle +} [n_0 h_\lambda + n_\mu k^{\mu}_{\cdot\lambda}]\, d\tau + \int_{\langle S\rangle -} [n_0 h_\lambda + n_\mu k^{\mu}_{\cdot\lambda}]\, d\tau = 0,$$

where $\langle S\rangle +$, $\langle S\rangle -$ indicate that the integrands are to be evaluated on the positive and negative sides of $\langle S\rangle$. Recalling that the normal on $\langle S\rangle +$ has the same sense as that of $'S_3$ and that the normal of $\langle S\rangle -$ has the same sense as that of $''S_3$, we may reduce the above relation to

$$\int_{\langle S\rangle} [n_0 \langle h_\lambda\rangle + n_\mu \langle k^{\mu}_{\cdot\lambda}\rangle]\, d\tau = 0, \tag{2.18}$$

where $\langle h_\lambda\rangle$, $\langle k^{\mu}_{\cdot\lambda}\rangle$ are the "jumps" in h_λ, $k^{\mu}_{\cdot\lambda}$. Finally, since the above integral is taken over an arbitrary part of $\langle S\rangle$, we may replace the integral relation by the point condition

$$\frac{\partial\phi}{\partial t} \langle h_\lambda\rangle + \frac{\partial\phi}{\partial x^\mu} \langle k^{\mu}_{\cdot\lambda}\rangle = 0. \tag{2.19}$$

The relation (2.19) is valid at each point of the discontinuity manifold $\langle S\rangle$.

For the cases when (2.12), (2.13) are valid, we obtain relations of type

$$\frac{\partial\phi}{\partial t}\langle h\rangle + \frac{\partial\phi}{\partial x^\mu}\langle k^\mu\rangle = 0. \tag{2.20}$$

III

DISCONTINUITIES IN THE DERIVATIVES OF v^λ, p, ρ: THE CHARACTERISTIC MANIFOLDS

In this case, we apply the formulas (2.19), (2.20) to the integrands of (2.14) through (2.16). We find the jumps in the derivatives satisfy the relations

$$\frac{\partial\phi}{\partial t}\langle\partial_j(\rho v_\lambda)\rangle + \frac{\partial\phi}{\partial x^\mu}\langle\partial_j(\rho v^\mu v_\lambda + p\delta^\mu_\lambda)\rangle = 0 \tag{3.1}$$

$$\frac{\partial\phi}{\partial t}\langle\partial_j\rho\rangle + \frac{\partial\phi}{\partial x^\mu}\langle\partial_j(\rho v^\mu)\rangle = 0 \tag{3.2}$$

$$\frac{\partial\phi}{\partial t}\langle\partial_j[\rho(\frac{q^2}{2} + e)]\rangle + \frac{\partial\phi}{\partial x^\mu}\langle\partial_j[\rho(\frac{q^2}{2} + h)v^\mu]\rangle = 0 \tag{3.3}$$

along the discontinuity manifold $\langle S\rangle$. In the above equations the vector

$$\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x^\mu}$$

represents the components of *any vector normal to the discontinuity manifold*. In the future work of this section we assume that q, p, ρ, v^λ, e, h, S are continuous but that the derivatives of these quantities may be discontinuous. To simplify the discussion, we shall ignore the discontinuities of the derivatives of the quantities e, h, S (the entropy) and hence consider only the first two equations of the above system.

Further, if we write the gas law as

$$p = p(S, \rho)$$

then

$$\partial_j p = c^2\partial_j\rho + \frac{\partial p}{\partial S}\partial_j S, \qquad c^2 = \left.\frac{\partial p}{\partial\rho}\right]_S$$

In view of our assumption, we may express the discontinuity in $\partial_j p$ as

$$\langle\partial_j p\rangle = c^2\langle\partial_j\rho\rangle.$$

First, we show that *no manifolds exist along which* $\partial_j p$, $\partial_j \rho$ *are discontinuous, but* $\partial_j v_\mu$ *are continuous*. This result follows immediately from the equations of motion. However, we shall develop this result by considering the first two equations of the above system. If we assume that $\partial_j v_\mu$ are continuous, then we may write

$$\partial_j \langle(\rho v^\lambda)\rangle = v^\lambda \langle \partial_j \rho\rangle, \qquad \partial_j \langle(\rho v^\mu v_\lambda)\rangle = v^\mu v_\lambda \langle \partial_j \rho\rangle,$$
$$\langle \partial_j (\delta^\mu_\lambda p)\rangle = \delta^\mu_\lambda \langle \partial_j p\rangle = \delta^\mu_\lambda c^2 \langle \partial_j \rho\rangle.$$

Through use of these relations the first two jump relations (3.1), (3.2) reduce to

$$\text{(3.4)} \qquad \langle \partial_j \rho\rangle \left[v_\lambda \left(\frac{\partial \phi}{\partial t} + v^\mu \frac{\partial \phi}{\partial x^\mu}\right) + c^2 \frac{\partial \phi}{\partial x^\lambda}\right] = 0$$

$$\langle \partial_j \rho\rangle \left[\frac{\partial \phi}{\partial t} + v^\mu \frac{\partial \phi}{\partial x^\mu}\right] = 0$$

These equations are compatible if and only if $\phi(x^\lambda, t)$ is the solution of the system:

$$\frac{\partial \phi}{\partial x^\mu} = 0, \qquad \frac{\partial \phi}{\partial t} + v^\mu \frac{\partial \phi}{\partial x^\mu} = 0.$$

Since this last system possesses the solutions

$$\phi_j(x^\lambda, t) = c_j, \qquad j = 0,1,2,3,$$

there exist only a finite number of points at which $\partial_j v_\mu$ are continuous but $\partial_j p$, $\partial_j \rho$ are discontinuous. From the second equation (3.4) one can obtain one conclusion which is valid even when e, h, S have discontinuous derivatives. Since this equation remains valid, we see that the discontinuity manifolds for which v_μ, ρ, p, S, e, h, $\partial_j v_\mu$ are continuous but $\partial_j \rho$, $\partial_j S$ are discontinuous belong to the hypersurfaces

$$\frac{\partial \phi}{\partial t} + v^\mu \frac{\partial \phi}{\partial x^\mu} = 0.$$

These hypersurfaces consist of families of stream lines. This follows from the fact that the equation implies that $d\phi/dt = 0$ as we follow the gas or fluid.[13]

13. The expression $d\phi/dt = \partial\phi/\partial t + v^\mu\, \partial\phi/\partial x^\mu$ is known as the material derivative. Its vanishing implies that one is following a fluid particle as one moves along ϕ = constant: H. Lamb, Hydrodynamics, Dover Publications, N.Y., 1945, p. 3.

Secondly, we shall state that discontinuity manifolds along which $\partial_j\rho$, $\partial_j p$ are continuous and $\partial_j v_\mu$ are discontinuous consist of stream lines. The verification of this result is left to the reader.

Finally, we consider the case where ρ, p, v_μ, etc., are continuous but possess discontinuous derivatives along $\langle S\rangle$. We shall show that *these discontinuities are propagated along the characteristic hypersurfaces which satisfy the partial differential equation*

$$(3.5)\qquad (\frac{\partial\phi}{\partial t})^2 + 2v^\mu \frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial t} + (v^\mu v^\lambda - c^2 g^{\lambda\mu})\frac{\partial\phi}{\partial x^\lambda}\frac{\partial\phi}{\partial x^\mu} = 0.$$

In this case, the jumps in $\partial_j\rho$, $\partial_j p$, $\partial_j v_\lambda$, satisfy the relations:

$$\langle\partial_j(\rho v_\lambda)\rangle = \rho\langle\partial_j v_\lambda\rangle + v_\lambda\langle\partial_j\rho\rangle$$

$$\langle\partial_j(\rho v^\mu v_\lambda)\rangle = \rho v_\lambda\langle(\partial_j v^\mu)\rangle + \rho v^\mu\langle(\partial_j v_\lambda)\rangle + v^\mu v_\lambda\langle(\partial_j\rho)\rangle$$

$$\langle\partial_j(\delta^\mu_\lambda p)\rangle = \delta^\mu_\lambda c^2\langle(\partial_j\rho)\rangle .$$

Substituting the above into the jump relations (3.1), (3.2), we obtain

$$\langle\partial_j\rho\rangle(v_\lambda\frac{\partial\phi}{\partial t} + v_\lambda v^\mu\frac{\partial\phi}{\partial x^\mu} + c^2\frac{\partial\phi}{\partial x^\lambda})$$

$$+\rho\langle\partial_j v_\lambda\rangle(\frac{\partial\phi}{\partial t} + v^\mu\frac{\partial\phi}{\partial x^\mu}) + \rho v_\lambda\frac{\partial\phi}{\partial x^\mu}\langle\partial_j v^\mu\rangle = 0,$$

$$\langle\partial_j\rho\rangle(\frac{\partial\phi}{\partial t} + v^\mu\frac{\partial\phi}{\partial x^\mu}) + \rho\frac{\partial\phi}{\partial x^\mu}\langle\partial_j v^\mu\rangle = 0.$$

If we solve the second equation for $\rho\frac{\partial\phi}{\partial x^\mu}\langle\partial_j v^\mu\rangle$ and substitute into the first, we obtain a simplified version of this first relation:

$$(3.6)\qquad \langle\partial_j\rho\rangle\, c^2\frac{\partial\phi}{\partial x^\lambda} + \rho\langle\partial_j v_\lambda\rangle(\frac{\partial\phi}{\partial t} + v^\mu\frac{\partial\phi}{\partial x^\mu}) = 0.$$

By eliminating $\langle\partial_j v_\lambda\rangle$ between (3.6) and the preceding equation, we obtain the characteristic equation (3.5), when $\langle\partial_j\rho\rangle \neq 0$. It is easily shown that, $\phi(x^\lambda, t) = $constant, cannot consist of stream lines.

These last two sets of equations consist of sixteen linear equations for the sixteen unknown "jumps", $\langle\partial_j\rho\rangle$, $\langle\partial_j v_\lambda\rangle$. However, the characteristic condition (3.6) implies that the determinant[14] of this system

14. The determinant is
$$(\frac{\partial\phi}{\partial t} + v^\alpha\frac{\partial\phi}{\partial x^\alpha})^8[(\frac{\partial\phi}{\partial t})^2 + 2v^\mu\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial t} + (v^\mu v^\lambda - c^2 g^{\lambda\mu}\frac{\partial\phi}{\partial x^\lambda}\frac{\partial\phi}{\partial x^\mu})]^4.$$

vanishes. Further, the original equations of motion furnish the four "jump" conditions:

$$\langle \frac{\partial v_\lambda}{\partial t}\rangle + v^\mu \langle \partial_\mu v_\lambda \rangle + \frac{c^2}{\rho} \langle \partial_\lambda \rho \rangle = 0,$$

$$\langle \frac{\partial \rho}{\partial t}\rangle + \rho g^{\lambda\mu} \langle \partial_\mu v_\lambda \rangle + v^\mu \langle \partial_\mu \rho \rangle = 0.$$

Solving (3.6) for $\langle \partial_j v_\lambda \rangle$, we obtain

$$\langle \partial_j v_\lambda \rangle = -c^2 \frac{\partial \phi}{\partial x^\lambda} \langle \partial_j \rho \rangle / \rho (\frac{\partial \phi}{\partial t} + v^\mu \frac{\partial \phi}{\partial x^\mu}).$$

By substituting this relation into the above four jump conditions, we obtain a system of four linear equations in the four unknowns, $\langle \partial_j \rho \rangle$. Upon simplification, these equations become

$$\frac{\partial \phi}{\partial x \lambda} \langle \frac{\partial \rho}{\partial t}\rangle + [v^\mu \frac{\partial \phi}{\partial x^\mu} - \delta^\mu_\lambda (\frac{\partial \phi}{\partial t} + v^\alpha \frac{\partial \phi}{\partial x^\alpha})] \langle \partial_\mu \rho \rangle = 0,$$

$$\frac{\partial \phi}{\partial t} \langle \frac{\partial \rho}{\partial t}\rangle + [2 v^\mu \frac{\partial \phi}{\partial t} + (v^\alpha v^\mu - c^2 g^{\alpha\mu}) \frac{\partial \phi}{\partial x^\alpha}] \langle \partial_\mu \rho \rangle = 0.$$

Evidently, the determinant[15] of this system of equations must vanish. We see that if

$$\langle \partial_j \rho \rangle = \lambda \, \partial_j \phi$$

where λ is a scalar, then the above jump conditions are satisfied. Further, the formula for $\langle \partial_j v_\lambda \rangle$ becomes

$$\langle \partial_j v_\lambda \rangle = -\lambda c^2 \frac{\partial \phi}{\partial x^\lambda} \partial_j \phi / \rho (\frac{\partial \phi}{\partial t} + v^\mu \frac{\partial \phi}{\partial x^\mu}). \tag{3.7}$$

This last formula is an extension of a well-known result for the steady state case.[16]

IV

DISCONTINUITIES OF v_λ, p, ρ, S, E, h: THE CONTACT AND SHOCK MANIFOLDS

We assume that: (1) *on each side* of the discontinuity manifold, the values of v_λ, p, ρ, S, e, h, and their space-time derivatives are

15. This determinant is
$$(\frac{\partial \phi}{\partial t} + v^\mu \frac{\partial \phi}{\partial x^\mu})^2 [(\frac{\partial \phi}{\partial t})^2 + 2 v^\alpha \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial t} + (v^\lambda v^\alpha - c^2 g^{\lambda\alpha}) \frac{\partial \phi}{\partial x^\lambda} \frac{\partial \phi}{\partial x^\alpha}] = 0.$$

16. R. Courant and D. Hilbert, loc. cit., Vol. II, p. 357.

continuous; (2) *in crossing* the discontinuity manifold the values of v_λ, p, ρ, S, e, h are discontinuous. If we apply the general formulas (2.19), (2.20) to the integrands of (2.11) through (2.13), we obtain the jump relations:

$$(4.1) \qquad \frac{\partial\phi}{\partial t}\langle \rho v_\lambda\rangle + \frac{\partial\phi}{\partial x^\mu}(\langle \rho\ v^\mu v_\lambda\rangle + \delta^\mu_\lambda\langle p\rangle\ ) = 0,$$

$$(4.2) \qquad \frac{\partial\phi}{\partial t}\ \langle\rho\rangle + \frac{\partial\phi}{\partial x^\mu}\ \langle\rho\ v^\mu\rangle\ = \ 0,$$

$$(4.3) \qquad \frac{\partial\phi}{\partial t}(\langle\frac{\rho q^2}{2}\rangle + \langle\rho e\rangle) + \frac{\partial\phi}{\partial x^\mu}(\langle\frac{\rho q^2 v^\mu}{2}\rangle + \langle\rho h v^\mu\rangle) = 0.$$

One may express the above system in terms of the values of ρ, v^λ, etc., *on one side of the discontinuity manifolds*, $\langle S\rangle$, and the jumps, $\langle\rho\rangle$, $\langle v^\lambda\rangle$, etc. To do this, we note that if the subscripts, 2 and 1, indicate the values of functions on the sides of $\langle S\rangle$ then

$$\langle\alpha\beta\rangle = (\alpha\beta)_2 - (\alpha\beta)_1 = (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) + \alpha_1\beta_2 + \alpha_2\beta_1 - 2\alpha_1\beta_1$$

Upon simplifying the right-hand side of the above we find that the jump may be expressed in the form

$$\langle\alpha\beta\rangle = \langle\alpha\rangle\ \langle\beta\rangle + \alpha_1\langle\beta\rangle + \beta_1\langle\alpha\rangle.$$

By use of a similar expansion we may write

$$\langle\alpha\beta\rangle = -\langle\alpha\rangle\ \langle\beta\rangle + \alpha_2\langle\beta\rangle + \beta_2\langle\alpha\rangle.$$

If we expand $\langle\rho v^\lambda\rangle$, $\langle\rho v^\lambda v_\mu\rangle$ in terms of $\langle\rho\rangle$, $\langle v^\lambda\rangle$, then the jump relations (4.1), (4.2) become a cubic and a quadratic in the jumps, $\langle\rho\rangle$, $\langle v^\lambda\rangle$, $\langle p\rangle$. Further, if (4.3) is expanded and the gas law is used, we obtain a fourth degree equation in the jumps. This remark is essential in understanding the difference between characteristic manifolds and manifolds of contact and shock discontinuity. From the jump relations of section III we see that *characteristic manifolds are determined by linear homogeneous equations in the unknown jumps*, $\langle\partial_j\rho\rangle$, $\langle\partial_j v_\lambda\rangle$, etc. However, *shock and contact manifolds are determined by non-linear homogeneous equations in the unknown jumps*, $\langle\rho\rangle$, $\langle v_\lambda\rangle$, *etc*. Due to the linear homogeneous property of the defining equations, one may easily determine the partial differential equation satisfied by the characteristic manifolds. This is obtained by requiring the determinant to vanish. Further, from the theory of linear homogeneous systems, we saw that if the characteristic manifold is a *hypersurface*, then the determinant of the equations in section 3 is at most of rank fifteen, and all the jumps are known in terms of one jump. On the other hand, the theory of a non-linear homogeneous algebraic system is diametrically opposed to that of the linear system. In the non-linear system, *no criterion (not involving the jumps) exists for determining the partial differential equation for the shock manifolds. (We shall show that the contact manifolds always consist of families of stream lines.)* However, if this shock manifold is known (also the values of ρ, p, v_λ on one side of it), then the jumps in v_λ, ρ, p, may be determined.

We initiate our study of the discontinuity system for shock and contact manifolds, (4.1) through (4.3), by writing the first equation of this system in the expanded form

$$[\langle\rho\rangle\langle v_\lambda\rangle + \rho_1\langle v_\lambda\rangle + \langle\rho\rangle(v_\lambda)_1]\frac{\partial\phi}{\partial t} + [\langle\rho v^\mu\rangle\langle v_\lambda\rangle + \langle\rho v^\mu\rangle(v_\lambda)_1$$

$$+ (\rho v^\mu)_1\langle v_\lambda\rangle]\frac{\partial\phi}{\partial x^\mu} + \langle p\rangle\frac{\partial\phi}{\partial x^\lambda} = 0.$$

Through use of the second jump relation (4.2), we may simplify the above equation and obtain

$$\rho_1\langle v_\lambda\rangle\left(\frac{\partial\phi}{\partial t} + v_1^\mu\frac{\partial\phi}{\partial x^\mu}\right) + \langle p\rangle\frac{\partial\phi}{\partial x^\lambda} = 0. \tag{4.4}$$

If we form the scalar product of this last equation and $e^{\alpha\beta\lambda}(\partial\phi/\partial x^\beta)$ where $e^{\alpha\beta\lambda}$ is the permutation tensor, we eliminate the unknown jump, $\langle p\rangle$, and obtain

$$\left(e^{\alpha\beta\lambda}\langle v_\lambda\rangle\frac{\partial\phi}{\partial x^\beta}\right)\left(\frac{\partial\phi}{\partial t} + v_1^\mu\frac{\partial\phi}{\partial x^\mu}\right) = 0.$$

The two possible solutions of the above are obtained by setting each factor equal to zero. *The corresponding discontinuities will be denoted as contact and shock discontinuities.* That is, along a *contact discontinuity,*

$$\frac{\partial\phi}{\partial t} + v_1^\mu\frac{\partial\phi}{\partial x^\mu} = 0; \tag{4.5}$$

and along a *shock discontinuity*

$$e^{\alpha\beta\lambda}\langle v_\lambda\rangle\frac{\partial\phi}{\partial x^\beta} = 0. \tag{4.6}$$

The *contact discontinuity* condition implies that *these discontinuity manifolds consist of families of stream lines.* By using the decomposition following (4.3), it is seen that instead of the contact condition (4.5) one may write

$$\frac{\partial\phi}{\partial t} + v_2^\mu \frac{\partial\phi}{\partial x^\mu} = 0.$$

On the other hand, the *shock discontinuity condition* implies that the *tangential component of velocity is continuous across a shock manifold.*

First, we consider the *contact discontinuities*. By expanding the second jump relation (4.2) and using the above contact condition, we find that along contact discontinuities

$$\langle v^\mu \rangle \frac{\partial\phi}{\partial x^\mu} = 0.$$

The above relation shows that *the normal component of velocity is continuous across a contact manifold*. Further, by use of the relations (4.4) and the above contact condition (4.5), we see that

$$\langle p \rangle = 0.$$

This means that the *pressure is continuous across a contact manifold.* From these results, we obtain the following conclusions as to the properties of contact manifolds:

(1) contact manifolds consist of families of stream lines;

(2) the normal component of velocity is continuous across a contact manifold; fluid mass is not transported across the discontinuity manifold;

(3) the tangential component of velocity is discontinuous across a contact manifold;

(4) the pressure is continuous across a contact manifold;

(5) the density is discontinuous (in general) across a contact manifold.

Since the contact manifold consists of stream lines, it follows that the normal component of velocity at the contact manifold is zero in the stationary case ($\frac{\partial\phi}{\partial t}=0$). Hence, for this case the velocity vector is tangent to the contact manifold. This type of discontinuity is equivalent to a vortex sheet.[17]

Secondly, we consider the *shock manifolds*. Since the tangential component of velocity is continuous across the shock manifold, we may write

$$\langle v_\lambda \rangle = \left(\langle v^\mu \rangle \frac{\partial\phi}{\partial x^\mu} \frac{\partial\phi}{\partial x^\lambda} \right) / \left(g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} \right)$$

17. L. M. Milne-Thomson, Theoretical Hydrodynamics, Macmillan Co., London, 1938, p. 339.

If we expand the second jump relation, (4.2), multiply the resulting equation by

$$\frac{\partial\phi}{\partial x^\lambda} \Big/ \left(g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta}\right)$$

and use the expression for $\langle v_\lambda \rangle$ derived in the preceding formula, we obtain the following relation between $\langle v_\lambda \rangle$ and $\langle \rho \rangle$

$$\langle v_\lambda \rangle = -\langle \rho \rangle \left(\frac{\partial\phi}{\partial t} + v_1^\mu \frac{\partial\phi}{\partial x^\mu}\right) \frac{\partial\phi}{\partial x^\lambda} \Big/ \left((\langle\rho\rangle + \rho_1) g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta}\right).$$

Further, the equation (4.4) shows that

$$\langle v_\lambda \rangle = -\langle p \rangle \frac{\partial\phi}{\partial x^\lambda} \Big/ \rho_1 \left(\frac{\partial\phi}{\partial t} + v_1^\mu \frac{\partial\phi}{\partial x^\mu}\right).$$

By equating the right-hand sides of the last two equations, we obtain the interesting relation

$$(4.7) \qquad (\langle\rho\rangle + \rho_1)\langle p \rangle g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} = \langle\rho\rangle \rho_1 \left(\frac{\partial\phi}{\partial t} + v_1^\mu \frac{\partial\phi}{\partial x^\mu}\right)^2.$$

This formula may be used in showing that small disturbances propagate along characteristic manifolds (see section 5). Further, this last relation is of some interest for the case of polytropic gases ($p = A\rho^\gamma$, A is a function of the entropy, $\gamma = 1.4$ for air at standard conditions). For such gases, it is easily shown that

$$\gamma p = \rho c^2.$$

Now, consider a shock manifold which separates a region of "quiet" ($v_1^\lambda = 0$, $\rho_1 =$ constant, $p_1 =$ constant, $c_1^2 = 0$) from a moving fluid. By use of the above gas law and the above conditions in the region of "quiet", we find that (4.7) reduces to

$$c^2(\langle\rho\rangle + \rho_1) g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} = \gamma \rho_1 \langle\rho\rangle \left(\frac{\partial\phi}{\partial t}\right)^2.$$

Thus, if c^2, $\langle\rho\rangle$ are known, then one may determine the shock manifold.

Due to the non-linear character of the jump equations (4.1) through (4.3), very little *general* theory exists for the shock manifolds. However, the one-dimensional shocks have been thoroughly studied. We refer the reader to the literature[18] for further details of this subject.

18. R. Courant and K. O. Friecrichs, loc, cit.

V

THE PROPAGATION OF SMALL DISTURBANCES: WAVE FRONTS

If we expand the jump relations (4.1), (4.2) by means of (4.4) and neglect the quadratic and cubic jumps, we obtain the relations

$$\rho_1\left(\frac{\partial\phi}{\partial t} + v_1^{\mu}\frac{\partial\phi}{\partial x^{\mu}}\right)\langle v_\lambda\rangle + \rho_1 (v_\lambda)_1 \langle v^{\mu}\rangle \frac{\partial\phi}{\partial x^{\mu}}$$

$$+ (v_\lambda)_1\left(\frac{\partial\phi}{\partial t} + v_1^{\mu}\frac{\partial\phi}{\partial x^{\mu}}\right)\langle\rho\rangle + \langle p\rangle\frac{\partial\phi}{\partial x^{\lambda}} = 0. \tag{5.1}$$

$$\rho_1\frac{\partial\phi}{\partial x^{\mu}}\langle v^{\mu}\rangle + \langle\rho\rangle\left(\frac{\partial\phi}{\partial t} + v_1^{\mu}\frac{\partial\phi}{\partial x^{\mu}}\right) = 0.$$

By use of the second relation (5.1) we may replace the first equation of the above system by the relation

$$\rho_1\left(\frac{\partial\phi}{\partial t} + v_1^{\mu}\frac{\partial\phi}{\partial x^{\mu}}\right)\langle v_\lambda\rangle + \frac{\partial\phi}{\partial x^{\lambda}}\langle p\rangle = 0. \tag{5.2}$$

The system (5.1), (5.2) consists of four linear equations in the five unknown jumps, $\langle v_\lambda\rangle$, $\langle p\rangle$, $\langle\rho\rangle$. Evidently, we must adjoin the jump relation (4.3) to this system. In order to avoid extensive computation, we shall assume that the entropy changes are negligible and

$$\langle p\rangle = c^2\langle\rho\rangle .$$

By solving (5.2) for $\langle v_\lambda\rangle$ and using this result to eliminate $\langle v_\lambda\rangle$ in (5.1) we obtain the characteristic equation (3.5). This result is often stated in the following form: *weak shocks are propagated along characteristic manifolds.*

At any given time, t_0, the characteristic discontinuity manifold

$$\phi(x^{\lambda}, t_0) = \text{constant} = \langle c\rangle,$$

$$t = t_0$$

is a surface in space-time. *Such a surface is called a wave front.* By varying t_0, $\langle c\rangle$, we obtain ∞^2 wave fronts. If we fix $\langle c\rangle$ and vary t_0, we determine those wave fronts which are the space-like cross sections of a given characteristic manifold at various times. Finally, if we fix t_0 and vary $\langle c\rangle$, we obtain the Mach surfaces. For the case of steady flows, the Mach surfaces coincide with the characteristic surfaces.[19]

We shall prove that $c = (dp/d\rho)^{1/2}$ is the velocity of the wave fronts on the characteristic manifolds determined by (3.5). Let us consider a fluid particle whose Lagrangian coordinates[20] are a^{λ}. By expressing the characteristic manifolds in the Lagrangian form

$$\phi[x^{\lambda}(a^{\mu}, t), t] = \langle c\rangle$$

we see that to each value of t, there corresponds a value of $\langle c\rangle$ and, hence, a wave front. Evidently,

$$\frac{d\phi}{dt} = \frac{d\langle c\rangle}{dt}$$

is a measure of the rate at which these wave fronts propagate. If we follow the fluid particle with coordinates a^{λ} on the wave fronts, we find that

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + v^{\lambda}\,\frac{\partial\phi}{\partial x^{\lambda}}$$

Squaring both sides of this equation and using the relation (3.5), we may express the above relation as

$$\left(\frac{d\phi}{dt}\right)^2 = c^2\left[g^{\alpha\beta}\,\frac{\partial\phi}{\partial x^{\alpha}}\,\frac{\partial\phi}{\partial x^{\beta}}\right]$$

The expression in the brackets in this last equation represents the magnitude of the x^{λ}-*space gradient of* ϕ. That is, if the arc length element along the x^{λ}-space orthogonal trajectories of the wave fronts, with $\langle c\rangle$ variable, is denoted by ds, then

$$g^{\lambda\mu}\,\frac{\partial\phi}{\partial x^{\lambda}}\,\frac{\partial\phi}{\partial x^{\mu}} = \left(\frac{d\phi}{ds}\right)^2.$$

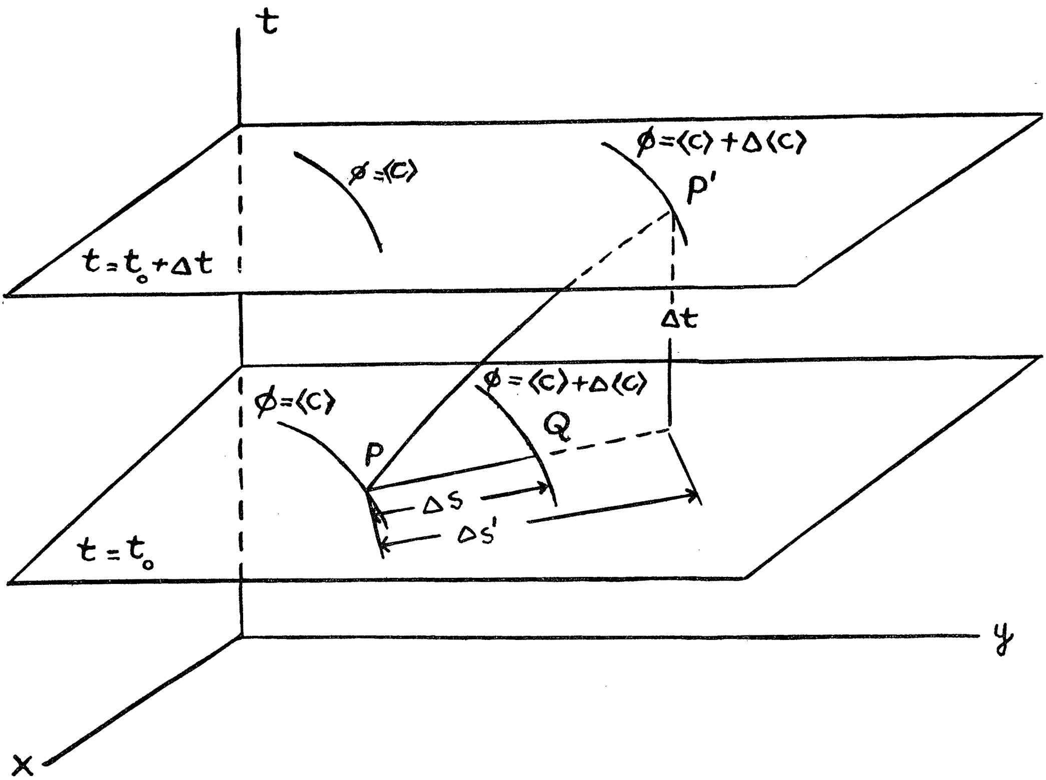
From these last two equations, we find that

$$\frac{ds}{dt} = c\,.$$

Thus, the scalar, c, is the ratio of the distance, perpendicular to the wave front, to the time for a particle moving in the front (see figure on following page). This ratio seems to be a reasonable measure of the local velocity of a wave front.

In the two-dimensional and time figure, a geometric picture of the above result is shown. The fluid particle is located at P on the wave front $\phi = \langle c\rangle$ at time $t = t_0$. At time $t = t_0 + \Delta t$, this fluid particle moves

20. See reference 13, p. 12.



The Velocity of Propagation, $c = \lim \frac{\Delta s}{\Delta t}$, in x, y, t space.

to $'P$, in the wave front, $\phi = \langle c \rangle + \Delta\langle c \rangle$. The element PQ is an orthogonal trajectory of the wave fronts

$$\begin{cases} \phi = \langle c \rangle \\ t = t_0 \end{cases} \qquad \begin{cases} \phi = \langle c \rangle + \Delta\langle c \rangle \\ t = t_0 \end{cases}$$

and Δs is the arc length along this orthogonal trajectory. One often says that "c is the local sound speed".

The following terms[21] are usually used in discussing wave propagation:

$$u = \frac{\partial \phi}{\partial t} \Big/ \left(g^{\lambda\mu} \frac{\partial \phi}{\partial x^\lambda} \frac{\partial \phi}{\partial x^\mu}\right)^{1/2} = \begin{cases} \text{velocity of displacement of wave} \\ \text{or velocity of wave front,} \end{cases}$$

$$c = \frac{d\phi}{dt} \Big/ \frac{d\phi}{ds} = \frac{d\phi}{dt} \Big/ \left(g^{\lambda\mu} \frac{\partial \phi}{\partial x^\lambda} \frac{\partial \phi}{\partial x^\mu}\right)^{1/2} = \begin{cases} \text{velocity of propagation of wave} \\ \text{or local velocity of sound,} \end{cases}$$

21. See reference 1, p. 105, formula [54].

$$q_n = v^{\lambda} n_{\lambda} = \frac{ds'}{dt} = \begin{cases} \text{velocity of a fluid particle on} \\ \text{the wave front in a direction} \\ \text{normal to the front.} \\ \text{(See the figure for the signifi-} \\ \text{cance of } \Delta s'\text{).} \end{cases}$$

From the above definitions and the formula

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + v^{\mu} \frac{\partial \phi}{\partial x^{\mu}}$$

it follows that

$$c = u + q_n .$$

University of Michigan

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

CLASSROOM SPECULATIONS ON A PARACHUTE PROBLEM

Roger Osborn

The following is a short discussion of a problem which created considerable interest in a differential equations class recently. Due to the nature of the problem, it could be used just as well in stimulating thought and discussion (and incidentally as an exercise in integration) in a course in integral calculus.

In Kells' ELEMENTARY DIFFERENTIAL EQUATIONS (3rd ed., McGraw-Hill, 1947) there is a footnote (p. 35) to a problem on the free fall of a human body before the opening of a parachute. The footnote takes its data from an article in the LITERARY DIGEST of November 10, 1928 (Vol. 99, no. 6, pp. 70, 72, 74). Briefly stated the incident reported in the LITERARY DIGEST was this: a parachutist weighing 180 lbs. jumped from an airplane and allowed himself to fall freely. By observation from the ground, it was learned that his velocity did not increase indefinitely, as had previously been supposed, but that it tended toward a limit. After dropping 1200 ft. in 11 seconds the parachutist's velocity was 173 ft./sec. and it remained nearly constant at that figure until he opened his parachute. When the parachute was opened, it caused his velocity to decrease to about 15 ft./sec. in a drop of 20 ft., after which his velocity again remained nearly constant at 15 ft./sec.

These data agree fairly well with the assumption that the resistance of air to his free fall is proportional to his velocity. The two questions raised in class, and answered by approximations satisfactory to all were these: (a) after the parachute was open, to what function of the velocity was the resistance of the parachute most nearly proportional (realizing that our facilities for performing integrations were limited), and: (b) to how many g's of stress was the parachutist subjected?

It was felt by some of the members of the class that the given data were sufficient to determine the answer to the first of these questions to a fair degree of accuracy, and that the power of the velocity suggested by the text might not be very accurate. It was decided to work the problem without reference to the text. Following is the scheme used by the class.

To answer the first question, the class assumed all factors other than the force of gravity and the resistance of the parachute to be negligible. Further it was assumed that the resistance (in pounds) of the parachute was proportional to some power of the velocity. Thus equations (1) and (1′) were obtained. (We assumed $g = 32.2$ ft./sec.^{2})

$$\frac{180}{32.2}\frac{dv}{dt} = 180 - \frac{180}{k^n}v^n \tag{1}$$

$$\frac{180}{32.2}v\frac{dv}{ds} = 180 - \frac{180}{k^n}v^n \tag{1'}$$

It was decided after some preliminary work that the most convenient form of the factor of proportionality is that used in (1) and (1′). Equation (1′) was solved by separation of variables as follows:

$$\int_{173}^{15}\frac{k^n v\,dv}{k^n - v^n} = \int_0^{20} 32.2\,ds = 644. \tag{2}$$

The integration of the left member of (2) was performed after assigning various values to n. Due to the physical implications of the problem, it was considered absurd to assign the value $n=1$ (which implies the resistance of the parachute is proportional to the velocity as was the resistance of the air before the parachute was opened), but in the following tabulation of values of n, the value $n=1$ is included for completeness. The integration of the left member of (2) gives rise to a transcendental equation in k which the class solved by approximate methods. It could be seen that if the acceleration were to approach zero at the end of the 20 ft. fall, the right member of equation (1) would have to be zero, which led the class to the conclusion that the numerical value of k would have to be very near 15. The following table shows the values of n assigned and the approximate roots obtained for the resulting equations in k.

n	1	3/2	2	3
k	4	10	13.5	14.99

It was felt by the class that other values of n could not be used profitably due to the difficulty of performing the integration in equation (2). From the above data the class concluded that the best approximation of the resistance of the parachute was that it is proportional to the cube of the velocity. (No attempt was made to make better approximations by subdividing the interval.)

The answer to the second question was more difficult to obtain due to the fact that the parachutist was subjected to a highly variable force. It was decided that a best approximation could be obtained by solving equation (1) with $n = 3$ and $k = 14.99$ to find the time for the parachutist to complete the 20 ft. fall. Letting t_1 be this time, the solution of equation (1) led to $t_1 = 0.54$ sec. Since the parachutist decelerated 158 ft./sec.^{2} in 0.54 sec., his average deceleration was 292.6 ft./sec.^{2} or about $9g$.

The University of Texas

NOTE ON CERTAIN ENVELOPES

Malcolm Foster

This note is concerned with the envelope of the family of lines which cut a given curve isosogonally.

1. *The equations of the envelope.* Let C, $y = f(x)$, be any given curve, and l the line of the family which intersects C at $P(x, y)$; and let θ be the constant angle of intersection. The slope of l is readily found to be

$$\frac{\alpha + y'}{1 - \alpha y'} \tag{1}$$

where $\alpha = \tan\theta$, and the equation of l is

$$(\alpha + y')(X - x) + (\alpha y' - 1)(Y - y) = 0 .$$

On differentiating this with respect to the parameter x, the parametric equations of the envelope γ of these lines are found to be

$$\begin{aligned} X &= x + \frac{\alpha(1 + y'^2)(1 - \alpha y')}{(1 + \alpha^2)y''}, \\ Y &= y + \frac{\alpha(1 + y'^2)(\alpha + y')}{(1 + \alpha^2)y''}. \end{aligned} \tag{2}$$

Let us denote this point of tangency (X, Y) of l and γ by Q. We find that the length of PQ is given by

$$PQ = \frac{\alpha(1 + y'^2)^{3/2}}{\sqrt{1 + \alpha^2}\, y''} . \tag{3}$$

Since this may be written $PQ = R\sin\theta$, where R is the radius of curvature of C at P, we have the theorem:

THEOREM 1. The points (2) for all values of θ lie on a circle through P whose center is on the normal at this point and whose diameter is equal to the radius of curvature.

It is evident that γ becomes the evolute of C when

$$\theta = \frac{\pi}{2},$$

and is identical with C when $\theta = 0$.

2. *The arc of* γ. Let us determine the length of the arc Q_1Q_2 of γ which corresponds to the arc of C from P_1 (x_1, y_1) to $P_2(x_2, y_2)$. From (1)

$$\frac{dY}{dX} = \frac{\alpha + y'}{1 - \alpha y'}; \tag{4}$$

and from (2)

$$dX = \frac{\alpha[y'y''(2 - 3\alpha y') - y'''(1 + y'^2)(1 - \alpha y')] + y''^2}{(1 + \alpha^2)y''^2}\, dx. \tag{5}$$

Hence

$$Q_1Q_2 = \int_{x_1}^{x_2} [1 + (\frac{dY}{dX})^2]^{1/2}\, dX,$$

or

$$Q_1Q_2 = \int_{x_1}^{x_2} \frac{(1+y'^2)^{1/2}[\alpha\{y'y''^2(2-3\alpha y')-y'''(1+y'^2)(1-\alpha y')\} + y''^2]}{(1-\alpha y')(1+\alpha^2)^{1/2}y''^2}dx \tag{6}$$

On differentiating (3)

$$\frac{d(PQ)}{dx} = \frac{\alpha(1 + y'^2)^{1/2}[3y'y''^2 - y'''(1 + y'^2)]}{\sqrt{1 + \alpha^2}\; y''^2}, \tag{7}$$

and on using (7) in (6) we have, after some reduction,

$$Q_1Q_2 = \int_{(x_1, y_1)}^{(x_2, y_2)} d(PQ) + \frac{1}{\sqrt{1 + \alpha^2}} \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx,$$

or

$$Q_1Q_2 = P_2Q_2 - P_1Q_1 + \frac{s}{\sqrt{1 + \alpha^2}},$$

where s = the arc of C from P_1 to P_2. This may also be written

(8) $$Q_1Q_2 = P_2Q_2 - P_1Q_1 + s\cos\theta = (R_2 - R_1)\sin\theta + s\cos\theta\ .$$

From the above we have:

THEOREM 2. The curves γ are rectifiable when, and only when, C is rectifiable.

It is readily seen from (3) and (8) that Q_1Q_2 becomes the arc of the evolute when

$$\theta = \frac{\pi}{2},$$

since in this case P_2Q_2 and P_1Q_1 become radii of curvature; and Q_1Q_2 reduces to the arc of C when $\theta = 0$. A very simple illustration of the use of (8) is the determination of the entire length of γ (which is a circle) when C is a circle of radius a. Since from (3) PQ is a constant, this length is $2\pi a\cos\theta$, where $a\cos\theta$ is the radius of γ. This, of course, is readily seen from elementary geometrical ideas.

We also point out that in the use of (8) we must limit our consideration of the arc of γ to those sections along which R (or PQ) either increases or decreases.

3. *The radius of curvature of γ.* On using (4) and (5) we find

$$\frac{d^2Y}{dX^2} = \frac{(1+\alpha^2)^2y''^3}{(1-\alpha y')^2[\alpha\{y'y''^2(2-3\alpha y')-y'''(1+y'^2)(1-\alpha y')\} + y''^2]}\ .$$

Hence the radius of curvature of γ is

(9) $$P = \frac{(1+y'^2)^{3/2}[\alpha\{y'y''^2(2-3\alpha y')-y'''(1+y'^2)(1-\alpha y')\} + y''^2]}{(1+\alpha^2)^{1/2}(1-\alpha y')y''^3}\ .$$

4. *An example.* Let us determine the curve γ when C is the cycloid

$$x = a(t - \sin t), \qquad y = a(1 - \cos t)\ .$$

On putting $y' = \sin t/(1 - \cos t)$, $y'' = -1/a(1 - \cos t)^2$ in (2) we have the following parametric equations of γ:

$$X = at + a\left[\frac{\alpha^2-1}{1+\alpha^2}\sin t + \frac{2\alpha}{1+\alpha^2}\cos t\right] - \frac{2a\alpha}{1+\alpha^2}\ ,$$

$$Y = a + a\left[\frac{\alpha^2 - 1}{1+\alpha^2}\cos t - \frac{2\alpha}{1+\alpha^2}\sin t\right] - \frac{2a\alpha^2}{1+\alpha^2}$$

Now $(\alpha^2 - 1)/(1 + \alpha^2)$ and $2\alpha/(1 + \alpha^2)$ are the cosine and sine respectively of some constant angle β. Hence these equations may be written

$$X + \frac{2a\alpha}{1+\alpha^2} = a\,[t + \sin(t + \beta)]\ ,$$

$$Y + a = a\,[1 + \cos(t + \beta)] + \frac{a(1-\alpha^2)}{1+\alpha^2},$$

where in the latter equation a has been added to each side. On setting $t + \beta = \varphi + \pi$, we have

$$X + \frac{2a\alpha}{1+\alpha^2} + a(\beta - \pi) = a(\varphi - \sin\varphi)\ ,$$

$$Y + \frac{2a\alpha^2}{1+\alpha^2} = a(1 - \cos\varphi)\ .$$

These equations become $\xi = a(\varphi - \sin\varphi)$, $\eta = a(1 - \cos\varphi)$ when the origin is moved to the point

$$\left[a(\pi - \beta) - \frac{2a\alpha}{1+\alpha^2}\ ,\quad - \frac{2a\alpha^2}{1+\alpha^2}\right]\ .$$

Hence we have:

THEOREM 3. The envelope of a family of lines which intersect a cycloid isogonally is a cycloid whose generating circle is the same as that of the given cycloid.

The cycloid which is the evolute of a cycloid represents the special case when $\theta = \pi/2$.

Finally, let us consider the locus, as θ assumes all values, of the point (h,k) to which the origin is moved in the above translation of axes. We have

$$h = a(\pi - \beta) - \frac{2a\alpha}{1+\alpha^2}\ ,\qquad k = -\frac{2a\alpha^2}{1+\alpha^2}$$

From the second of these relations

$$\alpha = \sqrt{-k}/\sqrt{k+2a}\ .$$

Also, put

$$\pi - \beta = \pi - \text{arc sin} \frac{2\alpha}{1+\alpha^2} = \lambda .$$

Hence

$$\sin \lambda = 2\alpha/(1 + \alpha^2) = \sqrt{-k^2 - 2ak}\ /a ,$$

and $h = a(\lambda - \sin \lambda)$. On squaring each side of the equation

$$a \sin \lambda = \sqrt{-k^2 - 2ak}$$

we have $a^2 \sin^2\lambda = -k^2 - 2ak$; and if a^2 be subtracted from each member it follows that $a^2\cos^2\lambda = (k + a)^2$. Consequently, $k = a(-1 + \cos \lambda)$. The locus of (h,k) is therefore another cycloid. It is generated by a point on a circle of radius a which rolls along the "under side" of the x-axis.

5. *Locus of centers of curvature of γ at Q.* Let T be the center of curvature of C at P. Then the normals to all γ's at the points Q pass through T; and the line QTW intersects the evolute of C at the same constant angle θ at which PQ intersects C. Consequently, from Theorem 1 we have

THEOREM 4. The locus of the centers of curvature of all γ's at Q is a circle of which a diameter is the radius of curvature of the evolute of C at T.

An analytical proof of this theorem is readily given with the use of (9).

Obviously, this theorem may be applied indefinitely. If V be the center of curvature of the evolute at T, the locus of the centers of curvature of the evolutes of the γ's is a second circle of which a diameter is the radius of curvature at V of the evolute of C.

6. *The involutes of γ.* These are the orthogonal trajectories of the family of lines l. From (1) the coordinates of any point T on l are

$$(10) \qquad \xi = x + \frac{(1 - \alpha y^1)t}{\sqrt{1+\alpha^2}\sqrt{1+y'^2}}, \qquad \eta = y + \frac{(\alpha + y^1)t}{\sqrt{1+\alpha^2}\sqrt{1+y'^2}},$$

where t is the distance along l from $P(x,y)$ to T. The condition that T be displaced orthogonally to l is evidently which, on using (10), readily reduces to

$$\sqrt{1 + y'^2}\,dx + \sqrt{1 + \alpha^2}\,dt = 0 \ ,$$

or

$$ds + \sqrt{1 + \alpha^2}\,dt = 0 \ .$$

Hence

$$t = \frac{c - s}{\sqrt{1 + \alpha^2}} = (c - s) \cos \theta \ , \tag{11}$$

where c is an arbitrary constant, and s is measured from some point on C.

The reader will find an interesting application of (11) in the example which follows Theorem 2, where C is a circle of radius a, γ is a circle of radius $a \cos \theta$ concentric with C, and the orthogonal trajectories of the lines l are involutes of the circle γ.

The results of this paper may also be readily obtained by the use of the moving trihedral of C.

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ROUND TABLE ON FERMAT'S LAST THEOREM

A Note on Fermat's Last Theorem

L. D. Grey

INTRODUCTION

This paper deals with the diophantine equation $X^{2p} + Y^{2p} = Z^{2p}$ where X, Y, and Z are integers, relatively prime in pairs, $\neq 0$ and p an odd prime. Solutions of this equation may be divided into two categories, (a) none of the unknowns divisible by p, (b) one of the unknowns divisible by p. Griselle[1] has shown that no solutions exist for (a) unless perhaps p is of the form $8A + 1$. We propose to extend this result to $24A + 1$ where $12A + 1$ contains no factor $\equiv 3 \pmod 4$, by showing that any solution of (a) requires p to be of the form $3A + 1$ where $3A + 2$ contains no factor $\equiv 3 \pmod 4$.

PROOF

The equation

(1) $$X^{2p} + Y^{2p} = Z^{2p}$$

may be written in the form

$$(X^p)^2 + (Y^p)^2 = (Z^p)^2.$$

Since the sum of two odd squares cannot be divisible by 4, Z must be odd and X and Y of different parity. Let Y be even.

From (1) we obtain

(2) $$Z^{2p} - X^{2p} = Y^{2p}$$

(3) $$(Z^p + X^p)(Z^p - X^p) = Y^{2p}.$$

Since Z and X are odd, $(Z^p + X^p)$ and $(Z^p - X^p)$ are even. Their greatest common factor is 2. The g.c.f. of these two expressions must divide their sum and difference, hence it must divide $2Z^p$ and $2X^p$. But Z^p and X^p are relatively prime since Z and X are relatively prime hence their g.c.f. $=2$. Let $Y = 2^t q$.

(4) (upper or lower signs hold) $$\frac{(Z^p \pm X^p)}{2} \frac{(Z^p \mp X^p)}{2^{2pt-1}} = q^{2p}.$$

1. Thomas Griselle, "Mathematics Magazine", Vol. 26, No. 5.

Since the two factors on the left are relatively prime each must be an integral $2p$-th power

$$(5) \qquad Z^p \pm X^p = 2e^{2p}$$

$$(6) \qquad Z^p \mp X^p = 2^{2pt-1} f^{2p}$$

and $g^{2p} = (ef)^{2p}$, $Y^{2p} = 2^{2pt} g^{2p}$. By a theorem due to Jaquemet[2] we know that $\frac{Z^p \pm X^p}{Z \pm X}$ and $Z \pm X$ are relatively prime when Z and X are relatively prime and $Z \pm X$ is not divisible by p. Hence we obtain fron (5) and (6)

$$(7) \qquad \frac{Z^p + X^p}{Z + X} = Z^{p-1} - Z^{p-2}X + Z^{p-3}X^2 + \cdots + X^{p-1} \quad (\text{to } p \text{ terms}) = h^{2p}$$

$$(8) \qquad \frac{Z^p - X^p}{Z - X} = Z^{p-1} + Z^{p-2}X + Z^{p-3}X^2 + \cdots + X^{p-1} \quad (\text{to } p \text{ terms}) = j^{2p}$$

where h and j are odd. h and j must also be relatively prime for otherwise (5) and (6) would have a common odd factor. If we add and subtract (5) and (6) we see that this factor must divide $2Z^p$ and $2X^p$ and hence Z^p and X^p but Z^p and X^p have a g.c.f. $= 1$. Adding (7) and (8) we obtain

$$(9) \qquad Z^{p-1} + Z^{p-3}X^2 + Z^{p-5}X^4 + \cdots + X^{p-1} \quad \left(\frac{p+1}{2}\right) \text{ terms} = \frac{h^{2p} + j^{2p}}{2}.$$

Letting

$$\frac{h^p + j^p}{2} = a \quad \text{and} \quad \frac{h^p - j^p}{2} = b$$

where a and b are co-prime and of different parity we have

$$(10) \qquad Z^{p-1} + Z^{p-3}X^2 + Z^{p-5}X^4 + \cdots + X^{p-1} = a^2 + b^2.$$

A well-known theorem of elementary number theory tells us that no number of the form $4b + 3$ can divide the sum of two relatively prime squares. In other words the congruence

$$U^2 + V^2 \equiv 0 \pmod{4b + 3}$$

has no solution if $(U, V) = 1$. Hence the right member of (10) has no factor $\equiv 3 \pmod 4$. Furthermore, the right member of (10) is of the form $4A + 1$ while the left member consists of $(p + 1)/2$ odd squares. Every odd square is $\equiv 1 \pmod 8$ because $(2b + 1)^2 = 4b(b + 1) + 1$ where b or $b + 1$ is $\equiv 0 \pmod 2$. Hence

$$(11a) \qquad (8k + 1)\left(\frac{p+1}{2}\right) = 4A + 1$$

2. L. J. Mordell, "Three Lectures on Fermat's Last Theorem", p. 8.

(11b) $$(8k+1)p+8k+1 = 8A+2$$

(11c) $$(8k+1)p = 8(A-k)+1$$

it follows that p must be of the form $8C+1$.

Suppose p is of the form $3Q+2$ then (10) becomes

(12) $$Z^{3Q+1}+Z^{3Q-1}X^2+Z^{3Q-3}X^4+\cdots+X^{3Q+1} \quad (\frac{3(Q+1)}{2}\text{ terms}) = a^2+b^2.$$

The left member of (12) has $Z^4+Z^2X^2+X^4$ as a factor and (12) may be written

(13) $$(Z^4+Z^2X^2+X^4)(Z^{3Q-3}+Z^{3Q-9}X^6+Z^{3Q-15}X^{12}+\cdots+X^{3Q-3}) = a^2+b^2$$

where the right factor has $(Q+1)/2$ terms (Q odd). The left factor on the left side of (13) is of the form $4b+3$ since every odd square $\equiv 1 \pmod 4$. But this factor cannot divide the right member of (13) hence (13) and therefore (1) is impossible for $p=3Q+2$. Letting $p=3Q+1$, since every prime $\neq 3$ is either of the form $3Q+2$ or $3Q+1$ (10) becomes

(14) $$Z^{3Q}+Z^{3Q-2}X^2+Z^{3Q-4}X^4+\cdots+X^{3Q} \quad (\frac{3Q+2}{2}\text{ terms}) = a^2+b^2 \quad (Q\text{ even}).$$

Suppose now that $3Q+2$ has a factor λ where $\lambda\equiv 3 \pmod 4$ then the left hand member of (14) has a factor $\equiv 3 \pmod 4$ since (14) can be written

(15) $$\begin{aligned}&(Z^{2\lambda-2}+Z^{2\lambda-4}X^2+Z^{2\lambda-6}X^4+\cdots+X^{2\lambda-2})\\ &\times(Z^{3Q-2\lambda+2}+Z^{3Q-4\lambda}X^{2\lambda+2}+\cdots+X^{3Q-2\lambda+2}) = a^2+b^2\end{aligned}$$

where the left factor of the left member of (15) has λ terms and the right factor $(3Q+2)/2\lambda$ terms. Since λ is $\equiv 3 \pmod 4$ the left factor consists of λ squares and hence is $\equiv 3 \pmod 4$. But a^2+b^2 has no factor $\equiv 3 \pmod 4$ hence (15) and therefore (1) is impossible for $p=3Q+1$ if $3Q+2$ has a factor $\equiv 3 \pmod 4$.

Since p must be of the form $3A+1$ and $8A+1$ it must be of the form $24A+1$ where $12A+1$ contains no factor $\equiv 3 \pmod 4$ if (1) has a solution.

Additional reference: L. E. Dickson, "History of the Theory of Numbers", Vol. 2.

New York University
Institute of Mathematical Sciences

H. J. A. Duparc and A. Van Wijngaarden, Amsterdam, have attacked the problem of determining a lower bound for x, y, z, by a different method from that reported in the Mathematics Magazine, Volume 27, Number 4, page 213. However, they restrict their work, as we did, to the "first case". See Nieuw Archief Voor Wiskunde, Derde Serie–Deel II–No. 1, Maart 1954. This article is in English.

Professor Vandiver and the Lehmers have proven Fermat's Last Theorem, for both cases, for all exponents less than 2000. See N. A. S. January, 1954.

Considerable material for this Round Table is arriving and being digested for future issues.

Your participation is invited.

Errata in *Some Introductory comments on Fermat's Last Theorem*, Vol. 27, 4, p. 213: p. 213, equation 1, change $n > 3$ to $n > 2$, middle of page 214, "values of y" should read "values of n", small caps in some of the exponents on p. 215-216 should be lower case like the other exponents, p. 216 equation 11 and following equation, write $x \leq$ for $x <$, also change Volume XIV to Volume XLV in 3rd paragraph.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

201. *Proposed by Leon Bankoff, Los Angeles, California.*

Solve the following cryptarithm. The first four convergents in the continued fraction expansion of $\sqrt{**}$ are $\frac{*}{*}$, $\frac{**}{**}$, $\frac{***}{***}$, $\frac{****}{****}$ where the asterisks are integers.

202. *Proposed by Chih-yi Wang, University of Minnesota.*

Find the coordinates of the center of curvature of

$$y = x^{x \sin x (\operatorname{arc} \cot x) \log x}$$

at the point (1,1).

203. *Proposed by Norman Anning, Alhambra, California.*

Prove that three of the intersections of $x^2 - y^2 + ax + by = 0$ and $x^2 + y^2 - a^2 - b^2 = 0$ trisect the circle through these three points.

204. *Proposed by C. W. Trigg, Los Angeles City College.*

In the triangle ABC let the feet of the median (m_a), of the internal angle bisector (t_a), of the cevian (p_a) to the contact point of the incircle with a, and of the cevian (q_a) to the contact point of the excircle relative to A with a be respectively A_m, A_t, A_p and A_q. Use similar notation for the corresponding lines to b and c.

1) Determine the relationship between the sides of the triangle if the following triads are to be concurrent: p_a, m_b, t_c at S; p_a, q_b, m_c at R; m_a, p_b, t_c at T; q_a, p_b, m_c at V.

2) Show that A_pB_q and A_qB_p are parallel to AB; C_tB_p and SV are parallel to BC; and C_tA_p and RT are parallel to AC.

205. *Proposed by V. Thebault, Tennie, Sarthe, France.*

The locus of the point, the sum of the products of whose distances from the pairs of opposite sides of a regular polygon of $2n$ sides is constant is a circle concentric with the polygon.

206. *Proposed by W. E. Byrne, Lexington, Virginia.*

If the symbol arc tan u is defined to be the angle (in radians) between $-\pi/2$ and $\pi/2$ whose tangent is u, determine for the interval $-\pi \leq x \leq +\pi$ the various values of the constant in the formula $2^{\gamma} = \alpha + \beta +$ const. where $\alpha = \text{arc tan } [(\sqrt{3}\tan x)/2]$, $\beta = \text{arc tan } [(\cos x)/3]$ and $\gamma = \text{arc tan } [(2 \tan x/2 + 1)/3]$.

207. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

If $a^2 + b^2 = c^2$ in a Pythagorean triangle whose sides are integers, solve for positive integers x, y and z the equation:

$$(a + b + x)^2 + (a + b + y)^2 = (a + b - z)^2 + (3a + 3b + 3c)^2.$$

SOLUTIONS

Late Solutions

175. *Tiny M. Morrow, University of North Carolina.*

178. *E. P. Starke, Rutgers University.*

A Mathematical Stew

181. [November 1953] *Proposed by C. W. Trigg, Los Angeles City College.*

Each word or phrase in the following semi-coherent story is a mutation of the name of a mathematician. Identify:

(1) BREAD LOVER, (2) HIRES GAL, (3) GAVE, (4) CHEF LIST FOR, (5) MEAL, (6) MAKE RUM, (7) RAW GIN, (8) GREEN NOG, (9) GLEAN, (10) GUY'S HEN, (11) DRIED TO, (12) OUR FIRE, (13) SO IN POT, (14) RUIN A CLAM, (15) STIR OUT SAND, (16) NO PEA, (17) ELK IN, (18) AY! BOIL, (19) SOUP IN OLLA, (20) SKIN COD, (21) USC CORN PIE, (22) HER COB, (23) IS LEFT, (24) NO HAM, MAC, (25) COP CAKE, (26) FUMED CAFE, (27) HAL SET, (28) HUT TABLE, (29) ATE THE SUET, (30) ATE LOT, SIR, (31) SETS O.K., (32) NICE MODES, (33) SEE BULGE, (34) MAR BELT, (35) TABLE RIM, (36) INK MARK, (37) THEREON, (38) SHAME.

Solution by Leon Bankoff, Los Angeles, California. (1) DE ROBERVAL (2) GLAISHER (3) VEGA (4) CHRISTOFFEL (5) LAME (6) KUMMER (A) (7) WARING

(8) GERGONNE (9) NAGLE (10) HUYGENS (11) DIDEROT (12) FOURIER (13) POINSOT (14) MACLAURIN (15) DINOSTRATUS (16) PEANO (17) KLEIN (18) BOLYAI (1() APOLLONIUS (20) DICKSON (21) COPERNICUS (22) BÔCHER (23) STIFEL (24) MACMAHON (25) PEACOCK (26) MAC DUFFEE (27) THALES (28) THEBAULT (29) THEAETETUS (30) ARISTOTLE (31) STOKES (32) NICOMEDES (33) LEBESGUE (34) LAMBERT (35) BELTRAMI (36) KIRKMAN (37) NOETHER (38) AHMES.

Although all of the above deserve an "*A*" in mathematics, Kummer is the only one who received this grade. Another explanation is that the stray "*A*" represents a vitamin that got lost in this culinary nightmare.

Addendum to the solution by the proposer:

Los Angeles smog, or to ME, MURK in KUMMER may have clouded my vision so that an interchange of vowels occurred. Whatever the cause, in (6), E., the German mathematician, was inadvertently replaced by A., the contemporaneous German cellist and composer — thereby striking the wrong note.

182. [November 1953] *Proposed by E. P. Starke, Rutgers University.*

a) Show that

$$f(a) = \sum_{j=1}^{\infty} \left[\frac{a}{j}\right] = \sum_{k=1}^{a} V(k)$$

Where the brackets indicate the greatest integer function and $v(k)$ is the number of divisors of k.

b) Is there any limit to the ratio, $f(a)/a$ as $a \to \infty$?

Solution by L. D. Grey, New York University. The divisors of k range in value from 1 to k. To the jth number in this sequence we assign a value $\alpha_j = 1$ if j is a divisor of k and $\alpha_j = 0$ if j is not a divisor of k. Therefore,

$$V(k) = \sum_{j=1}^{k} \alpha_j$$

$$\sum_{j=1}^{a} V(k) = \sum_{j=1}^{a} (n_j)(\alpha_j) \qquad \begin{array}{l}(\alpha_j = 1 \text{ if } j \text{ is a} \\ \text{divisor of any of} \\ \text{the terms } 1, \cdots, a)\end{array}$$

where n_j denotes the number of terms in the sequence $1, 2, \cdots, a$ having j as a divisor. Since the only terms in the sequence $1, 2, 3, \cdots, a$ which are divisible by j are the jth terms, the number of multiples of j contained in the sequence is equal to $[\frac{a}{j}]$. Hence

$$\sum_{j=1}^{a} V(k) = \sum_{j=1}^{a} (n_j)(\alpha_j) = \sum_{j=1}^{a} \left[\frac{a}{j}\right]$$

$$= \sum_{j=1}^{\infty} \left[\frac{a}{j}\right] \quad \text{since} \quad \left[\frac{a}{j}\right] = 0 \quad \text{if} \quad j > a.$$

b) We shall show that there is no limit to the ratio $f(a)/a$ as $a \to \infty$.

We prove first that $\frac{a-j}{j} \le [\frac{a}{j}]$ for $a \ge j$. Let $a = pj + q \quad 0 \le q < j$. Then

$$[\frac{a}{j}] = p$$

$$\frac{a}{j} = p + \frac{q}{j}$$

$$\frac{a}{j} - 1 = p + \frac{q}{j} - 1$$

$$\frac{a-j}{j} = p - 1 + \frac{q}{j}$$

$$\frac{a-j}{j} = [\frac{a}{j}] + \frac{q}{j} - 1 \quad 0 \le q < j, \quad a \ge j$$

$$[\frac{a}{j}] - \frac{a-j}{j} = \text{a negative quantity}$$

Hence, each term in the series $\sum_{j=1}^{a} \frac{a-j}{ja}$ is less than the corresponding term of the series $\sum_{j=1}^{a} [\frac{a}{j}]/a$. The terms in the first series are $(1 - \frac{1}{a}) + (\frac{1}{2} - \frac{1}{a}) + (\frac{1}{3} - \frac{1}{a}) + \cdots$. The limit of this sequence is the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is known to be divergent as $a \to \infty$. Since each term of the latter series is larger than the corresponding term in the former series the latter series must also be a divergent one.

Also solved by *R. K. Guy, University of Malaya, Singapore.*

A Triangle Construction

183. [November 1953] *Proposed by D. L. Mac Kay, Manchester Depot, Vermont.*

Construct a triangle ABC having given, in position, the circumcenter O, the foot D of the altitude from A and the point of intersection U of the bisector of angle A with the side BC.

Solution by Leon Bankoff, Los Angeles California. The circle (U,UD) cuts the circle on diameter OU in E and E'. The tangent at D is cut by OE in A and by OE' in A'. UD extended cuts the circle (O,OA) in B and C, and the circle (O,OA') in B' and C'. ABC and $A'B'C'$ are the two triangles that may, in general, be constructed.

If $UD=UO$, A is found by the intersection of the tangents at O and at D, yielding a unique construction.

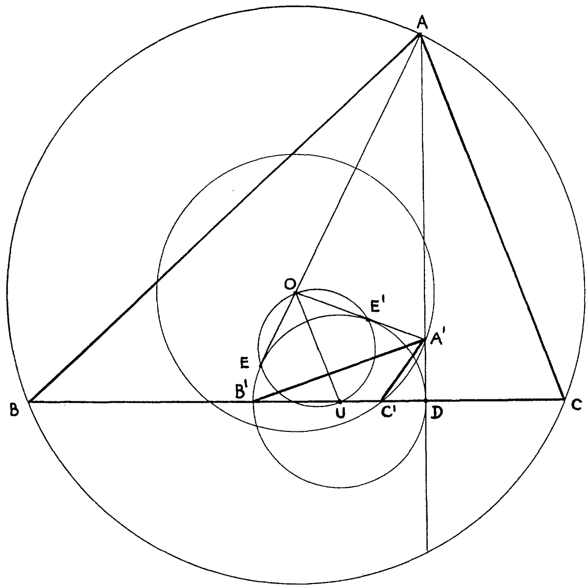
PROOF: By construction, O is the circumcenter and D the foot of the altitude, h_a, for the triangles ABC and $A'B'C'$. Since the right triangles AUD and AEU are congruent, AU is the bisector of angle EAD. Now AD and AO are known to be isogonal with respect to angle BAC. Hence AU is also the bisector of angle BAC. Similarly, UA' is the bisector of angle $B'A'C'$.

When $UD = UO$, E and E' coincide at O, and the same proof applies.

REMARKS: No solutions exist for $UD > UO$, or for $UD = UO$ when U, D and O are collinear.

When U, D, and O are collinear and $UD < UO$, we obtain two congruent right triangles with a common hypotenuse.

When O is distinct and U and D coincide, the solution is indeterminate or non-existent depending on whether OA is arbitrarily taken $>$ or $<$ OU. If O, U and D coincide, the solution is an infinitude of isosceles right triangles with O the midpoint of the hypotenuse.



Also solved by *Henry Gerhardt, Mobile, Alabama; John Jones Jr., Mississippi Southern College; A. Sisk, Maryville College; P. D. Thomas, Eglin Air Force Base, Florida* and the proposer.

The Sequence $\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}$

184. [November 1953] *Proposed by T. F. Mulcrone, Spring Hill College.*

Show that in the sequence $1/n$, $2/n$, $3/n$, $\cdots$, $(n-1)/n$, where n is a positive integer greater than 2, an even number of the terms are fractions in lowest terms.

I. *Solution by Richard K. Guy, University of Malaya, Singapore.* In the sequence $1/n$, $2/n$, $3/n$, ... , $(n-1)/n$ the number of fractions in lowest terms is exactly $\phi(n)$, Euler's totient function. $\phi(n) = n\Pi[1 - 1/p_i]$ where the p_i are the prime factors of n. For $n > 2$, $\phi(n)$ is even, thus there are an even number of fractions in the sequence in lowest terms.

II. *Solution by Norman Anning, Alhambra, California.* The theorem can be proved by showing that if k/n is a fraction in lowest terms so also is $1-k/n$. Thus fractions in lowest terms can be arranged in matched pairs. The theorem follows.

Also solved by *L. D. Grey, University of New York; Nathaniel Grossman, Aurora, Illinois; Abraham L. Epstein, Cambridge Research Center, Massachusetts; John Jones Jr., Mississippi Southern College; Lawrence A. Ringenberg, Eastern Illinois State College; L. Van Deventer, Eastern Illinois State College; and the proposer.*

An Endless Game

185. [November 1953] *Proposed by Francis L. Miksa, Aurora, Illinois.*

Four players, A, B, C, D, sit around a circular table. Each man has a number of matches in front of him. They play a match game as follows. First A removes enough matches from his pile and gives to his three other friends enough matches to multiply their holdings by 2, next B removes enough matches from his pile and gives to the three other players enough matches to multiply their holdings by factor 3, continuing C uses factor 4, D uses factor 5, A uses factor 6, and lastly B uses factor 7.

After those six plays it is found that each man has exactly the same amount of matches he started with at the beginning of the game.

What is the smallest number of matches each man could have at the beginning of the game? Develop a formula for N men, n plays, $n > N$, and n different factors.

Solution by John M. Howell, Los Angeles City College. Let the four players A, B, C and D each start with a, b, c and d matches respectively. Also let $t = a + b + c + d$. The table below indicates the number of matches held by each player after k plays.

k	A	B	C	D
0	a	b	c	d
1	$2a-t$	$2b$	$2c$	$2d$
2	$3!a-(1\cdot 3)t$	$3!b-2t$	$3!c$	$3!d$
3	$4!a-(1\cdot 3\cdot 4)t$	$4!b-(2\cdot 4)t$	$4!c-3t$	$4!d$
4	$5!a-(1\cdot 3\cdot 4\cdot 5)t$	$5!b-(2\cdot 4\cdot 5)t$	$5!c-(3\cdot 5)t$	$5!d-4t$
5	$6!a-(1\cdot 3\cdot 4\cdot 5\cdot 6+5)t$	$6!b-(2\cdot 4\cdot 5\cdot 6)t$	$6!c-(3\cdot 5\cdot 6)t$	$6!d-(4\cdot 6)t$
6	$7!a-(1\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7+5+7)t$	$7!b-(2\cdot 4\cdot 5\cdot 6\cdot 7+6)t$	$7!c-(3\cdot 5\cdot 6\cdot 7)t$	$7!d-(4\cdot 6\cdot 7)t$

To satisfy the rules the amount each player has for $k=6$ must equal what he started with so:

$$a = \frac{(1\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7+5+7)}{7!-1}t = \frac{2555}{5039}t$$

$$b = \frac{(2\cdot 4\cdot 5\cdot 6\cdot 7+6)}{7!-1}t = \frac{1686}{5039}t$$

$$c = \frac{(3\cdot 5\cdot 6\cdot 7)}{7!-1}t = \frac{630}{5039}t$$

$$d = \frac{(4\cdot 6\cdot 7)}{7!-1}t = \frac{168}{5039}t.$$

Since 5039 is prime, take $t = 5039$. This leads to

$$a = 2555, \quad b = 1686, \quad c = 630, \quad d = 168.$$

If there are N players and n plays where $n > N$, then the number held by the i th player at the start is given by:

$$g(i) = [i(i+2)(i+3)\cdots(n+1)] + [(i+N)(i+N+2)(i+N+3)\cdots(n+1)] + [(i+2N)(i+2N+2)(i+2N+3)\cdots(n+1)\ \cdots .$$

That is:

$$g(i) = \sum_{x=0}^{\frac{n-i}{N}}(i+xN)\prod_{y=1}^{n-i-xN}(i+xN+y+1).$$

It is interesting to note that $\sum_{i=1}^{N} g(i) = (n+1)! - 1$.

Also solved by *Monte Dernham, San Francisco, California; Abraham L. Epstein, Cambridge Research Center, Massachusetts; Sam Kravitz, East Cleveland, Ohio, and the proposer.*

A Binomial Identity

186. [November 1953] *Proposed by Edwin C. Gras, U. S. Naval Academy.*

Prove the following identity involving binomial coefficients:

$$\sum_{r=0}^{k}(-1)^r\, C_{k-1}^{n+r}\, C_r^k = 0.$$

I. *Solution by Chih-yi Wang, University of Minnesota.* Following C. Jordan, *Calculus of Finite Differences*, formulas on pages 68 and 133, we have:

$$\sum_{r=0}^{k}(-1)^r\, C_{k-1}^{n+r}\, C_r^k = (-1)^{n+k+1}\sum_{r=0}^{k} C_{-n-r-1}^{-k}\cdot C_r^k = (-1)^{n+k+1}\, C_{-n-1}^{0}$$

which is equal to 0 for all integral n except $n=-1$. The given sum is equal to $(-1)^k$ for $n=-1$.

II. *Solution by John M. Howell, Los Angeles City College.* Let

$$S(n,k) = \sum_{r=0}^{k}(-1)^r\binom{n+r}{k-1}\binom{k}{r}, \qquad n > k-2.$$

Now
$$S(n,k) = \sum_{r=0}^{k}(-1)^r \binom{k}{r} \sum_{z=k-2}^{n+r+1}\binom{z}{k-2},$$
since
$$\sum_{s=m}^{n}\binom{s}{m} = \binom{n+1}{m+1}.$$
So
$$S(n,k) = \sum_{r=0}^{k}\sum_{z=k-2}^{n+r+1}(-1)^r\binom{k}{r}\binom{z}{k-2},$$

Next interchanging the order of summation:
$$S(n,k) = \sum_{z=k-2}^{n+k+1}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\binom{z}{k-2} - \sum_{z=n+2}^{n+k+1}\sum_{r=0}^{z-n-2}(-1)^r\binom{k}{r}\binom{z}{k-2}.$$

The first term is zero as $\sum_{r=0}^{k}(-1)^r\binom{k}{r} = 0$. Also since
$$\sum_{x=0}^{a}(-1)^x\binom{b}{x} = (-1)^a\binom{b-1}{a} \quad \text{for } a < b,$$
we have
$$S(n,k) = -\sum_{z=n+2}^{n+k+1}\binom{z}{k-2}(-1)^{z-n-2}\binom{k-1}{z-n-2}.$$

Let $y = z - n - 2$. Then
$$S(n,k) = -\sum_{y=0}^{k-1}(-1)^y\binom{n+y+2}{k-2}\binom{k-1}{y} = -S(n+2,k-1).$$

Applying this transformation $k-1$ times
$$S(n,k) = (-1)^{k-1}S(n+2k-2,1).$$
However
$$S(m,1) = \binom{m}{0}\binom{1}{0} - \binom{m+1}{0}\binom{1}{1} = 0,$$

Therefore $S(n,k) = 0$.

Also solved by the proposer using Gamma functions.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and source, if known.

Q110. If a, b and c are sides of a triangle such that $a^2 + b^2 + c^2 = ab + bc + ca$, prove that the triangle is equilateral. [*Submitted by M. S. Klamkin.*]

Q111. Does the equation $2^{2^a} - 2^{2^b} = 10^4 C$ have a solution in integers a, b and c where $a \neq b$? [*Submitted by Charles Brenner.*]

Q112. A billiard ball is hit (without any "English") so that it returns to its starting point after hitting four different cushions. Show that the distance travelled by the ball is the same regardless of the starting point. [*Submitted by M. S. Klamkin.*]

Q113. A ball is dropped from a height of 10 feet. It rebounds one half the distance on each bounce. What is the total distance it travels? [*Submitted by J. M. Howell.*]

Q114. Sum $\sum_{n=1}^{\infty} \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \right] x^n$ [*Submitted by M. S. Klamkin.*]

Q115. Factor $a^{15} + 1$. [*Submitted by C. W. Trigg.*]

Q116. N men throw their hats in a ring then each one takes one at random. Those getting their own hats leave and the rest throw their hats in the ring again. What is the expected number of trials until all get their own hats? [*Submitted by D. L. Silverman.*]

ANSWERS

A110. The equation is equivalent to $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$. Thus $a = b = c$.

A111. Yes. 2^{2^x} takes on infinitely many values but finitely many (mod 10^4).

A112. Imagine that the four cushions are mirrors and we are using a light ray. By the principle of images it is easily shown that the distance equals twice the diagonal.

A113. If the height is h, after the first drop it rises $\frac{1}{2}h$, next $\frac{1}{4}h$ etc. The distance up and down then is twice $\frac{1}{2}h + \frac{1}{4}h + \cdots$. The first drop is h feet so the total is $3h$ or 30 feet.

A114. If $F(x) = \sum a_n x^n$ then $F(x)/(1-x) = \sum (a_1 + a_2 + \cdots + a_n) x^n$. Thus $F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$, so our sum is $S = \frac{e^x - 1}{1-x}$.

A115. $a^{15} + 1 = (a+1)(a^2 - a + 1)(a^{12} - a^9 + a^6 - a^3 + 1) = (a+1)(a^4 - a^3 + a^2 - a + 1)(a^{10} - a^5 + 1)$. Inspection reveals that $a^2 - a + 1$ does not divide $a^4 - a^3 + a^2 - a + 1$, hence it must divide $a^{10} - a^5 + 1$. So we write $a^{10} - a^5 + 1 = (a^{10} - a^9 + a^8)(a^9 - a^8 + a^7) - (a^7 - a^6 + a^5) - (a^6 - a^5 + a^4) - (a^5 - a^4 + a^3)(a^3 - a^2 + a)(a^2 - a + 1)$. Therefore we have: $a^{15} + 1 = (a + 1)(a^4 - a^3 + a^2 - a + 1)(a^2 - a + 1)(a^8 + a^7 - a^5 - a^4 - a^3 + a + 1)$.

A116. The expected number of matches per trial is 1, therefore the expected number of trials for all to match is n.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Reply to Mrs. Bernice Brown's comment on "Systems of Equations, Matrices, and Determinants", Vol. 27, No. 1, Sept.-Oct., 1953, p. 43.

First of all, the part of the article discussed by Mrs. Brown occurs under the heading "Rounding off errors." Obviously the solution to any (compatible) system of linear equations can be obtained exactly by a true elimination process (or any other "rational" process). The problem of "condition" arises whenever we use a "practical" elimination process (working to a fixed number of decimals) or, again, if we consider the effect of small changes in the coefficients, or in the right hand sides of the equation.

Second, let us try to describe the contents of the section. We do not attempt to define ill-condition — we give examples to show symptoms of this malady, and indicate measures (normal values of which are indicated), which, like pulse or temperature, may be helpful in diagnosis; we do not describe any remedies. We have suggested by an example that a system which exhibits the phenomenon when we attempt to solve it by means of one process, is also likely to behave badly when we try to solve it by another process.

John Todd

Theory of Equations. By Cyrus Colton MacDuffee, John Wiley and Sons, Inc., 440 Fourth Avenue, New York, 1954, 120 pages, $3.75.

Emphasizing the theory of polynomials, Cyrus Colton MacDuffee's *Theory of Equations* otherwise follows the outlines of a standard course. The new book, designed for junior and senior students, was published in February by John Wiley & Sons.

A concise and balanced treatment of the subject, the volume covers linear systems, rational solutions, and polynomials. Real and complex roots, relations among roots, and systems of higher degrees are the other topics covered.

Richard Cook

First Course in Calculus. By Professor Hollis R. Cooley, John Wiley & Sons, Inc., 440 Fourth Ave., New York, 643 pages, $6.00.

Designed for students, the new book stresses the understanding of concepts rather than formal proofs. Dr. Cooley has unified his subject around the tangent problem and the area problem. In the first nine chapters, he provides a rounded short course, including differentiation, inverse differentiation, and integration of algebraic functions, with applications.

Other important points include a full treatment of improper integrals and approximate integration, a discussion of the uniform convergence of power series, definition of the logarithmic function by an integral, and comparison of the orders of transcendental functions without dependence on L'Hospital's rule.

Richard Cook

A First Course in Ordinary Differential Equations. By Rudolph E. Langer, John Wiley & Sons, Inc., 440 Fourth Ave., New York, 1954, 249 pages, $4.50.

Written for the student with a year's experience in calculus, the new book presents the differential equation dually as a mathematical concept and as a technological tool. The author emphasizes differential equations of the first and second orders, with procedures for equations of the higher order given briefly.

An outstanding authority in the field, Dr. Langer is professor of mathematics at the University of Wisconsin. He was formerly editor of the Bulletin of the American Mathematical Society and is currently editor of the Duke Mathematical Journal.

Richard Cook

Introductory College Mathematics. By Adele Leonhardy, John Wiley & Sons, Inc., 440 Fourth Ave., New York, 459 pages, $4.90.

This book is designed to develop those mathematical concepts and techniques that are valuable in a general education program, and to present mathematics itself as one of the areas of general education.

By providing a broader treatment than is ordinarily given in the traditional first year course, the volume applies equally to students concentrating on this field, and those who do not specialize in mathematics or related subjects. The phases of mathematics utilized in humanities, social science, and natural science courses have been carefully checked by the author, who then selected varied material that would be suitable for different abilities, interests, and mathematical backgrounds.

Among the topics covered are the algebra of numbers, numbers in exponential form, measurement and computation, the comparison of quantities, functional relationships, variation, and the rate of change of a function. The chapter headings also include exponential and logarithmic functions, periodic functions, and simple statistical methods.

Dr. Leonhardy is chairman of the mathematics department at Stephens College.

Richard Cook

Theory and Applications of Distance Geometry. By Leonard M. Blumenthal, Oxford University Press, 1953, 11 + 347 pages, $10.00.

This book is a very welcome contribution to the field of mathematics. It contains for the first time in English an account of the general theory of abstract metric spaces or metric topilogy and a detailed study of metric methods in Euclidean, spherical, elliptic, and hyperbolic spaces. Applications are given to determinant theory, linear inequalities, and lattice theory. The aim of this book is to serve as a textbook for the advanced graduate student and also as a reference work for the specialists in the various fields of mathematics where metric methods are used. It contains a varied list of exercises, some of which may serve as the bases for master's theses. The references at the end of each chapter are provided so that the student may consult the original sources and progress in research on the particular subject matter. The bibliography is on pages 339-343, and the index is on pages 344-347.

The contents of this treatise on distance geometry is divided into four parts. Part I consists of a general introduction to abstract metric spaces, topology, a detailed study of abstract metric segments and lines, and the abstract theory of rectifiable curves and curvature. In Part II, are developed the metric and vector characterizations of Euclidean and Hilbert spaces. Part III is the study of the Non-Euclidean spaces including such subjects as spherical space, pseudo-spherical sets, elliptic and hyperbolic spaces. Finally in Part IV, applications are given to determinant theory, linear inequalities, and lattice theory. In all, there are fifteen chapters.

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Part I on Metric Spaces begins with a preliminary introduction to such subjects as Hausdorff topological spaces, metric and semi-metric spaces with examples, and lattice theory. Various characterizations of metric betweenness are given. Many theorems of Menger on metric segments and lines are studied. After a comprehensive study of rectifiable metric arcs, it is shown that in a finitely compact metric space, if two distinct points can be joined by a rectifiable arc, then they can be joined by a geodesic arc. The three definitions of curvature due to Menger, Alt, and Haantjes, are compared and contrasted. It is proved that a rectifiable are of a ptolemaic metric space is a metric segment if and only if the Menger curvature vanishes at all points of the arc. Part I concludes with an abstraction of torsion and Wald's metrization of Gaussian curvature.

In studying the priblem in Parts II and III of congruently imbedding an m-tuple of pairwise distinct points of a semi-metric space S is a given semi-metric space R, certain square and symmetric matrices called *distance matrices*, whose elements depend upon the distances $p_i p_j$ between every two points p_i and p_j of the m-tuple, play a very important role. For Euclidean n-space E_n, the elements r_{ij} for $i, j = 0, 1, \cdots, m$, of such a matrix obey the conditions: $r_{ii} = 0$; $r_{0j} = r_{j0} = 1$ for $j = 1, \cdots, m$; and all other $r_{ij} = (p_i p_j)^2$. Its determinant D_{m+1} is termed the Cayley-Menger determinant. For spherical n-space $S_{n,r}$, the elements of such a matrix are $r_{ij} = \cos p_i p_j / r$ for $i, j, = 1, \cdots, m$. The elements of such a matrix is denoted by Δ_m. For a given m-tuple congruently imbeddable in elliptic n-space $E_{n,r}$ there is an associated class of matrices $(\varepsilon_{ij} r_{ij})$ where (r_{ij}) is the matrix of the corresponding spherical m-tuple and (ε_{ij}) is an ε-matrix with elements $\varepsilon_{ii} = 1$, $\varepsilon_{ij} = \pm 1$. The determinant of any such matrix is denoted by Δ^*. For generalized hyperbolic n-space $H^{\phi}_{n,r}$, the elements of such a matrix are $r_{ij} = \phi(x_{ij}; r)$ where ϕ is a real single valued function of x and r with $\phi(0;r) \geq 0$, and $x_{ij} = p_i p_j$. Its determinant is denoted by Λ_m. For ordinary hyperbolic n-space $H_{n,r}$, $\phi(x;n) = \cosh x/r$. In the chapter of Part IV on Applications of Metric Methods to Determinant Theory, theorems concerning the ranks and forms of the determinants D_{m+1}, Δ_m, Δ^*_m, Λ_m, and some of their subclasses are obtained by means of the imbedding theory of m-tuples in the corresponding metric spaces.

One of the most fundamental concepts in this book is that of *congruence indices* (n,k) of a semi-metric space R relative to a class $\{S\}$ of semi-metric spaces. This signifies that any space S of the class, containing more than $n+k$ pairwise distinct points, is congruently imbeddable in R whenever each n of its points have that property. According as $k=0$ or 1, R has *congruence order* n or *quasi-congruence order* n It is proved that E_n, $S_{n,r}$, $H_{n,r}$, have the congruence indices $(n+3,0)$ relative to $\{S\}$, but $E_{n,r}$ does not possess this property. One of the more important peculiarities of elliptic space is that two subsets can be congruent without necessarily being superposable (the congruence may not be necessarily extendable to the whole space). For this reason, there is introduced the notion of *superposibility order* σ of a semi-metric space R, which is analogous to that of congruence order n. This means that every two subsets of R for which there is a one-to-one correspondence f such that correspinding σ-tuples of the two subsets are f-superposable, are themselves f-superposable. Thus E_n, $S_{n,r}$, $H_{n,r}$, have minimum superposibility order 2, whereas $E_{n,r}$ has minimum superposibility order $n+1$. A few of the theorems in Parts II and III involving these concepts may be mentioned. Every compact metric space has a finite or hyperfinite congruence order relative to the class of all separable semi-metric spaces. A connected, simply connected closure of a bounded open set of E_2, is a circular disk if and only if it has the congruence order 3 relative to the subsets of E_2. A hemisphere and a spherical cap of $S_{n,r}$ have the best congruence indices $(2n+1,1)$ and $(n+2,n)$ relative to $\{S\}$. The elliptic plane $E_{2,r}$ has congruence order 8 relative to $\{S\}$. In $E_{2,r}$, a cross, (the locus of points equidistant from two fixed distinct points), has congruence indices $(6,2)$ and congruence order 7 relative to $\{S\}$.

Many characterizations of Euclidean space E_n and Hilbert space H within the class $\{S\}$ of semi-metric spaces are given in Part II. For example, S is congruent to E_n if and only if (a) S is complete, convex, externally convex, (b) S has the weak four-point Euclidean property, (c) $D_{n+3}=0$ for every $(n+2)$-tuple of S, and (d) n is the smallest integer for which (c) holds. A separable semi-metric space S is congruently imbeddable in H if and only if for every k, D_{k+2} of each $(k+1)$-tuple of S either vanishes or else has the same sign as $(-1)^{k+1}$. A semi-metric space S is congruent to H if and only if (a) S is separable, complete, convex, externally convex, (b) S has the weak four-point Euclidean property, and (c) for every k, there is a $(k+1)$-tuple of S for which $D_{k+2}\neq 0$. The property that the Pythagorean theorem holds may replace the weak four-point Euclidean property. For every n, pseudo-E_n's exist, can not contain more than $n+3$ distinct points, and none of these is congruently imbeddable in H. A quasi-inner product space Σ obeying the Schwarz and existence postulates is a generalized Euclidean space. Also Σ has the weak four-point Euclidean property. This Σ is congruent to H if it is separable, complete, and for every k, a k-tuple of Σ exists for which the grammian $G_k\neq 0$. Finally Σ is congruent to E_n if and only if it is complete and n is the smallest integer for which the grammian G_{n+1} of every $(n+1)$-tuple of Σ vanishes.

An extension of $S_{n,r}$ is the semi-metric space Σ_r for which (a) the distance of every two points is $\leq \pi r$, (b) Σ_r has the weak four-point spherical property, and (c) Σ_r is complete, convex, and diametrized. It is shown that Σ_r is congruent to $S_{n,r}$ if and only if (a) there is an $(n+1)$-tuple of Σ_r for which $\Delta_{n+1}\neq 0$, and (b) for every $(n+2)$-tuple of Σ_r, $\Delta_{n+1}=0$. After listing five basic properties of $S_{n,r}$ and giving some of their logical consequences, pseudo-$S_{k,r}$'s are studied. Pseudo-$S_{k,r}$ sets of arbitrarily high power ≥ 2, exist. Among the many theorems on the intersections of convex sets in $S_{n,r}$, may be mentioned the following result which is an analogue of a theorem of Helly for Euclidean space. If every $n+1$ of the sets of a family of convex sets of $S_{n,r}$, the diameter of each member being less than $2\pi r/3$, have a common point, then the intersection of the whole family is not null.

To simplify the discussion of the imbedding and characterization theorems for $E_{n,r}$, a space E_r is defined which is congruent with a elliptic space of a finite or infinite number of dimensions. This E_r contains at least two distinct points and is a complete, convex semi-metric space of diameter $\pi r/2$ such that (I) if a quintuplet contains two linear triples and if a Δ^* formed from three of these points, one of which is common to the linear triples, is negative, they the principal minors of a Δ^* for the quintuplet, are non-negative, and (II) if p, q are distinct points with $pq < \pi r/2$, points $d(p)$, $d(q)$ exist such that q is between p and $d(p)$, p is between q and $d(q)$, and $pd(p) = qd(q) = \pi r/2$. To discuss the congruence of E_r to a given $E_{n,r}$, there are introduced two additional properties (one global and the other a localization of it), either one of which when added to the two properties of E_r, gives rise to a space congruent to the given $E_{n,r}$. The linear subspaces of E_r are found to be congruent to $E_{k,r}$ for $k = 1, 2, \cdots$. Any orthocentric quadruple of $E_{2,r}$, is freely moveable, and five points of a cross of $E_{2,r}$, are freely movable if no four of the points form a proper orthocentric quadruple. One of the more important results is the crowding theorem which states that every set of 8 points of $E_{2,r}$, contains a triple with perimeter $< \pi r$, and 8 is the smallest number with this property. It may be mentioned that metric bases for $E_{2,r}$, are not necessarily congruently invariant, but if a metric basis for $E_{2,r}$, contains more than five distinct points, it is a congruence invariant.

In the final chapter of Part III, generalized hyperbolic space $H^{\phi}_{n,r}$ is defined as a semi-metric space S with the properties: (I) for every $(n+2)$-tuple of S, $\Lambda_{n+2} = 0$, (II) there exists an $(n+1)$-tuple of S for which $\Lambda_{k+1} = 0$, and (III) if Λ_{k+1} of a $(k+1)$-tuple of S is not zero, then it has the sign of (-1) . Generalized hyperbolic space $H^{\phi}_{n,r}$, contains for each $k \leq n$, a generalized hyperbolic space $H^{\phi}_{k,r}$, determined by $k+1$ independent points $(\Lambda_{k+1} \neq 0)$ of $H^{\phi}_{n,r}$. By adjoining a fourth property to the set of three properties defining $H^{\phi}_{n,r}$, it can be shown that any two k-dimensional hyperbolic subspaces of $H^{\phi}_{n,r}$, are congruent. Also congruence indices of $H^{\phi}_{k,r}$, and pseudo-$H^{\phi}_{k,r}$'s are studied.

In Part IV, generalizations of certain theorems of Minkowski are given on a system of linear inequalities: $f_1(t)x_1 + \cdots + f_{n+1}(t)x_{n+1} \geq 0$, in the $n+1$ real indeterminates $x_1, \cdots, x_{n+1}$, where the $n+1$ functions $f_1(t), \cdots, f_{n+1}(t)$, are real, single valued, and not all zero for any t of a non-vacuous set T of real numbers. This system is finite (the Minkowski case), denumerable, or non-denumerable according to the power of T. The coefficients of any one of the inequalities of the system can be regarded as a set of direction numbers of a ray in E_{n+1} with initial point at the origin, and therefore the system of inequalities is in one-to one correspondence with a set C of points on the unit sphere $S_{n,1}$ with center at origin. Similarly the set of non-trivial solutions, which may be vacuous, can be visualized as a set of open rays in E_{n+1} with the deleted initial point at the origin, and hence this set of rays is in one-to-one correspondence with a certain set Σ of points on $S_{n,1}$. It is assumed that C is not congruently imbeddable in any $S_{k,1}$ where $k<n$, so that the system of inequalities is irreducible. A knowledge of the set functional $\Sigma(C)$ provides the total solution of the given system of inequalities. By applications of various metric theorems for subsets of spherical space, many theorems concerning $\Sigma(C)$ may be established. For example, as a consequence of the theorem that if S is any closed convex subset of $S_{n,1}$, and if S contains no diametrical point pairs, then $S = T^*(S)$, that is S is the convex extension of its set of terminal points, it is shown that $\Sigma(C) = T^*(\Sigma(C))$. Among the existence theorems established, may be stated the proposition that a system has a non-trivial solution if and only if each subsystem of $n+k+1$ members has a non-trivial solution in common with a fixed arbitrarily selected subsystem of $n-k+1$ linearly independent members of the system where $k = 0, 1, \cdots, n, n+1$. Also strict inequalities are considered.

The final subject in Part IV is concerned with metric methods in lattice theory. In a metric space, let $B(a,c,b)$ signify the fact that either c is between a and b, or else c coincides with a or b. Glivenko's lattice characterization of metric betweenness in a normed lattice L, is that $B(a,c,b)$ in the metric space $D(L)$ of L if and only if $ac+cb = c = (a+c)(c+b)$. A normed lattice L is distributive if and only if the necessary and sufficient condition for $B(a,c,b)$ is that $ab<c<a+b$ in L. Pitcher and Smiley adopted Glivenko's characterization of metric betweenness in a normed lattice L as the definition of lattice betweenness $LB(a,c,b)$ in an arbitrary lattice L. Lattice betweenness $LB(a,c,b)$ has many properties of metric betweenness. The theorem of Aronszajn that every normed lattice may be congruently imbedded in a convex normed lattice is proved. The work of Glivenko, Smiley, and Transue concerning the properties of the metric space $D(L)$ of a normed lattice L is given. The book closes with a discussion of autometrized Boolean algebras where the distance is an element of the given set.

The material of this book is presented in a very straightforward and elegant manner. The many concepts and theorems are logically deduced from various postulational systems so that no extraneous knowledge of mathematics is necessary for its comprehension. Although the author states that a knowledge of topology or abstract algebra would be helpful in the understanding of the subject matter, it seems to the reviewer that only mathematical maturity is a sufficient prerequisite. It is hoped in agreement with the author that parts of this book will be offered in graduate schools because of its usefulness and the manifold applications in related fields.

John De Cicco